# Invariance-Based Approach: General Methods and Pavement Engineering Case Study

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**Abstract**

In many application areas such as pavement engineering, the phenomena are complex, and as a result, we do not have first-principle models describing the corresponding dependencies. Luckily, in many such areas, there is a lot of empirical data and, based on this data, many useful empirical dependencies have been found. The problem is that since many of these dependencies do not have a theoretical explanation, practitioners are often hesitant to use them: there have been many cases when an empirical formula stops being valid when circumstances change. To make the corresponding empirical formulas more reliable, it is therefore desirable to look for theoretical foundations of these formulas. In this paper, we show that many of such dependencies can be naturally explained by using invariances. We illustrate this approach on the example of pavement engineering, but the approach is very general, and can be applied to other systems as well.

## 1 Formulation of the Problem

**Often, we do not have solid from-first-principles models.** In many application areas such as pavement engineering, the phenomena are complex, and as a result, we do not have first-principle models describing the corresponding dependencies.

**Instead, we have empirical models.** Luckily, in many such areas, there is a lot of empirical data and, based on this data, many useful empirical dependencies have been found.

**Problem: without a theoretical justification, an empirical model is often not perceived as reliable.** The problem is that many of these dependencies do not have a theoretical explanation, practitioners are often hesitant to use them: there have been many cases when an empirical formula stops being valid when obstacles change.

Even the great Newton naively believed that, since the price of a certain stock was growing exponentially for some time, it will continue growing – so he invested all his money in that stock and lost almost everything when the bubble collapsed.

To make the corresponding empirical formulas more reliable, it is therefore desirable to look for theoretical foundations of these formulas.

**What we do in this paper.** In this paper, we show that many of such dependencies can be naturally explained by using invariances.

**Case study.** In this paper, we illustrate this approach on the example of pavement engineering. However, this approach is very general, and can be applied to other systems as well; see, e.g., [33].

**Structure of the paper.** In Sections 2 and 3, we describe what are the basic invariances, what are the dependencies explained by these invariances, and how these dependencies can be combined. In Section 4, we list problems related to pavement engineering, review what empirical formulas have been found in this application area, and we show how general invariance ideas can provide theoretical justification for these formulas.

## 2 Basic Invariances and Related Dependencies

**Simplest case.** Let us start with the simplest case when we want to the dependence $y = f(x)$ between two physical quantities $x$ and $y$.

**Difference between mathematical and physical descriptions.** From the purely mathematical viewpoint, the problem seems straightforward: we need to find the relation between the two numerical values. However, from the physical viewpoint, we need to take into account that the same physical quantity can be represented by different numerical values: the specific value depends on what measuring unit we select (and, for some quantities, on the starting point).

**Numerical values of physical quantities depend on the measuring unit.** Let us start with the fact that we can use different measuring units. For example, we can measure distances in meters or kilometers; the same distance will be represented by different numbers: 2 km becomes 2000 m. In general, if we replace the original measuring unit with a different unit which is $\lambda > 0$ times smaller, all numerical values get multiplied by $\lambda$: $x \mapsto x' = \lambda \cdot x$.

**Notion of scale-scale (sc-sc) invariance.** In many physical situations, there is no selected measuring unit, so the formulas should not depend on what measuring unit we use.

Of course, we cannot simply require that the formula remains exactly the same if we change the unit for $x$: that would mean that

$$f(x) = f(\lambda \cdot x)$$

for all $\lambda$ and $x$ – and thus, that $f(x) = \text{const}$, i.e., that there is no dependence at all. In reality, if we change the unit for $x$, we need to appropriately change the unit for $y$. For example, the formula $y = x^2$ for the area $y$ of a square with side $x$ remains valid if we switch from meters to centimeters – but then we need to also change the measuring unit for area from square meters to square centimeters.

So, the desired property takes the following form: for each $\lambda > 0$, there should exist a value $\mu > 0$ such that:

- if $y = f(x)$,
- then $y' = f(x')$, where $x' \stackrel{\text{def}}{=} \lambda \cdot x$ and $y' \stackrel{\text{def}}{=} \mu \cdot y$.

This property is known as *scale-scale-invariance.*

*Terminological comment.* In physics, an invariance – in particular, scale-scale invariance – is often also called *symmetry*; see, e.g., [20, 54]. This terminology comes from the fact that what we usually called symmetry is actually invariance with respect to geometric transformations; physicists use this term to non-geometric invariance as well.

**Which dependencies are scale-scale-invariant.** Substituting $y' = \mu \cdot y$ and $x' = \lambda \cdot x$ into the formula $y' = f(x')$, we get $\mu \cdot y = f(\lambda \cdot x)$. Here, we have $y = f(x)$, so $f(\lambda \cdot x) = \mu \cdot f(x)$. Taking into account that $\mu$ depends on $\lambda$, we get the following expression:

$$f(\lambda \cdot x) = \mu(\lambda) \cdot f(x). \tag{1}$$

Small changes in $x$ should cause proportionally small changes in $y$, so the dependence $f(x)$ must be smooth (differentiable). It is known (see, e.g., [1]), that every smooth function $f(x)$ that satisfy the functional equation (1) for some $\mu(\lambda)$ has the power law form $f(x) = c \cdot x^a$. Vice versa, it is easy to show that every power law function has the scale-scale invariance property.

*Comments.*

- Actually, smoothness is not necessary: it is sufficient to require that the dependence $y = f(x)$ is measurable; see, e.g., [1].

- The general proof – for the case of measurable functions – is rather complicated, but for the smooth case, this result can be proven by using simple calculus. For completeness – and to explain to interested readers how such results are proven – we reproduce this known derivation in the Appendix.

**What if $y$ depends on several variables?** Similarly, for the dependence $y = f(x_1, \ldots, x_v)$ on several quantities $x_1, \ldots, x_v$, we could similarly require that for all possible tuples $(\lambda_1, \ldots, \lambda_v)$, there should exist a value $\mu(\lambda_1, \ldots, \lambda_v)$ such that if we have

$$y = f(x_1, \ldots, x_v), \tag{2}$$

then in the new units

$$x'_i = \lambda_i \cdot x_i \tag{3}$$

and

$$y' = \mu(\lambda_1, \ldots, \lambda_v) \cdot y, \tag{4}$$

we should have

$$y' = f(x'_1, \ldots, x'_v). \tag{5}$$

If we plug in the expressions (3) and (4) into the formula (5), we get

$$\mu(\lambda_1, \ldots, \lambda_v) \cdot y = f(\lambda_1 \cdot x_1, \ldots, \lambda_v \cdot x_v). \tag{6}$$

If we now plug in the expression for $y$ from formula (2) into this formula, we will conclude that

$$\mu(\lambda_1, \ldots, \lambda_v) \cdot f(x) = f(\lambda_1 \cdot x_1, \ldots, \lambda_v \cdot x_v). \tag{7}$$

It is known (see, e.g., [1]) that every measurable solution to this functional equation has the form

$$y = C \cdot x_1^{m_1} \cdot \ldots \cdot x_v^{m_v}. \tag{8}$$

**Sometimes, we can also select different starting points.** Scale-scale-invariance assumes that we have a fixed starting point for measuring a quantity. This is true for most physical quantities, but for some physical quantities, we can select different starting points. For example, for measuring temperature, we can select, as a starting point, the temperature at which water freezes – and get the usual Celsius scale – or we can select the absolute zero and thus get the Kelvin scale. For different purposes, different starting points may be more appropriate.

If we change a starting point for measuring $x$ to a different starting point which is $x_0$ units smaller the original one, then this value $x_0$ will be added to all numerical values of this quantity: $x \mapsto x' = x + x_0$, so that $x = x' - x_0$. Similarly, if we change a starting point for measuring $y$ to a different starting point which is $y_0$ units smaller than the original one, then this value $y_0$ will be added to all numerical values of this quantity: $y \mapsto y' = y + y_0$, so that $y = y' - y_0$.

**Notion of shift-shift (sh-sh) invariance.** In many physical situations, there is no selected starting point. So, it is reasonable to require that the formulas describing the dependence between $x$ and $y$ should not depend on what starting point we use. Of course, we cannot simply require that these formulas remains exactly the same if we change the starting point for $x$: that would means that $f(x) = f(x + x_0)$ for all $x_0$ and $x$ – and thus, that $f(x) = \text{const}$, i.e., that there is no dependence at all. In reality, if we change the starting point for $x$, we need to appropriately change the starting point for $y$.

So, the desired property takes the following form: for each $x_0$, there should exist a value $y_0$ such that:

- if $y = f(x)$,
- then $y' = f(x')$, where $x' \stackrel{\text{def}}{=} x + x_0$ and $y' \stackrel{\text{def}}{=} y + y_0$.

This property is known as *shift-shift-invariance.*

**Which dependencies are shift-shift-invariant?** Substituting $y' = y + y_0$ and $x' = x + x_0$ into the formula $y' = f(x')$, we get $y + y_0 = f(x + x_0)$. Here, we have $y = f(x)$, so $f(x + x_0) = f(x) + y_0$. Taking into account that $y_0$ depends on $x_0$, we get the following expression:

$$f(x + x_0) = f(x) + y_0(x_0). \tag{9}$$

Small changes in $x$ should cause proportionally small changes in $y$, so the dependence $f(x)$ must be smooth (differentiable). It is known that for smooth functions, the functional equation (9) implies that the function $f(x)$ must be linear: $f = a \cdot x + C$ [1]. Vice versa, it is easy to show that every linear function has the shift-shift invariance property.

*Comment.* Similarly to the case of scale-scale invariance, this result remains true if instead of smoothness, we require that the function $f(x)$ is measurable. The known proof for the smooth case is reproduced in the Appendix.

**Shift-to-scaling (sh-sc).** Let us consider the case when the dependence remains the same after we apply shift to $x$ and scaling to $y$. In this case, for every $x_0$, there exists a value $\mu(x_0)$ (depending on $x_0$) such that:

- if $y = f(x)$,
- then we have $Y = f(X)$, where $X = x + x_0$ and $Y = \mu(x_0) \cdot y$.

If we plug in the expressions for $Y$ in terms of $y$ and $X$ in terms of $x$ into the formula $Y = f(X)$, we conclude that $f(x + x_0) = \mu(x_0) \cdot y$. Here, $y = f(x)$, so we conclude that

$$f(x + x_0) = \mu(x_0) \cdot f(x). \tag{10}$$

It is known (see, e.g., [1]) that every measurable dependence $f(x)$ with this property has the form

$$f(x) = A \cdot \exp(a \cdot x), \tag{11}$$

for some $A$ and $a$.

Vice versa, it is easy to show that every linear function has the shift-to-scaling invariance property.

*Comment.* In the smooth case, this result is easy to prove; see Appendix.

**Scaling-to-shift (sc-sh).** Let us now consider the case when the dependence remains the same after we apply scaling to $x$ and shift to $y$. In precise terms, we assume that for every $\lambda > 0$, there exists a value $y_0(\lambda)$ (depending on $\lambda$) such that:

- if $y = f(x)$,
- then $Y = f(X)$, where $X = \lambda \cdot x$ and $Y = y + y_0(\lambda)$.

If we plug in the expressions for $Y$ in terms of $y$ and $X$ in terms of $x$ into the formula $Y = f(X)$, we conclude that $f(\lambda \cdot x) = y + y_0(\lambda)$. Here, $y = f(x)$, so we conclude that

$$f(\lambda \cdot x) = f(x) + y_0(\lambda). \tag{12}$$

It is known (see, e.g., [1]) that every measurable dependence $f(x)$ with this property has the form

$$f(x) = a \cdot \ln(x) + C, \tag{13}$$

for some $a$ and $C$.

Vice versa, it is easy to show that every linear function has the shift-to-scaling invariance property.

*Comment.* In the smooth case, this result is easy to prove; see Appendix.

**Which parameter values are more probable: general idea.** All the above families have some parameters that need to be determined: for example, the power law dependence $y = c \cdot x^a$ has two parameters: $c$ and $a$. A natural question is: which values of these parameters are more probable?

Our answer – that we will explain in this subsection – is that the most probable values are the extreme values, i.e., the values that correspodn gto the extreme points of the set of possible values. This conclusion is motivated by a result about the following de-noising problem – the result that was proven in the 1980s by B. S. Tsirel'son [55]; see also [34, 53].

In this de-noising problem, the sender's message ("signal") is described by several real values $\vec{s} = (s_1, \ldots, s_d)$. For an audio signal, these values $s_i$ are sound intensities at different moments of time. For a video signal, these are intensity values at different spatial points at different moments of time. In both cases, the value $d$ is large.

Usually, not all tuples of real numbers represent possible signals: there are bounds on each value, there are limitations on how much a real signal can change with time, etc. Let us denote the set of possible signals $\vec{s}$ by $A \subset \mathbb{R}^d$.

In practice, during transmission, noise is usually added the signal. So, the values that we receive describe this "signal + noise" combination

$$\vec{o} = (o_1, \ldots, o_d) = (s_1 + n_1, \ldots, s_d + n_d),$$

where $n_i$ denotes the (unknown) values of the noise. The problem is to reconstruct the signal $\vec{s}$ from the observed signal-plus-noise combination $\vec{o}$.

Usually, the noise is the joint effect of many independence factors. Thus, each noise value $n_i$ is the sum of a large number $N$ of small independent random variables. It is known that the distribution of such a sum is close to Gaussian – this follows from the *Central Limit Theorem* (see, e.g., [51]) according to which this distribution tends to Gaussian when $N \to \infty$. So, for all practical purposes, we can assume that the values $n_i$ are normally distributed, with some standard deviation $\sigma$. To reconstruct the signal $\vec{s}$ from the noisy signal $\vec{o}$, we can use the Maximum Likelihood approach – the standard statistical approach in which, crudely speaking, we select the most probable answer; see, e.g., [51]. For the normal distribution, the probability is proportional to $\exp\left(-\dfrac{1}{2\sigma^2} \cdot \sum\limits_{i=1}^{d} n_i^2\right)$. Thus, maximizing this probability is equivalent to minimizing the sum $\sum\limits_{i=1}^{d} n_i^2 = \sum\limits_{i=1}^{d} (o_i - s_i)^2$. This sum is the square of the distance between the actual (unknown) signal $\vec{s}$ and the observed noisy signal $\vec{o}$. So, minimizing the above sum is equivalent to minimizing this distance. Thus, this de-noising problem can be describing in geometric terms: given a tuple $\vec{o}$, find the closest tuple $\vec{s}$ in the set $A$ of possible signals.

Tsirel'son proved that the reconstructed signal is often an extreme point of the set $A$. Let us describe the main idea of Tsirel'son explanations as presented in [55].

Due to the Law of Large Numbers (see, e.g., [51]), for large $d$, the the average value of $(n_i)^2$ is close to the standard deviation $\sigma^2$, where $\sigma$ is the standard deviation of the noise. In other words, we can conclude that

$$(n_1)^2 + \ldots + (n_d)^2 \approx d \cdot \sigma^2.$$

In geometric terms, this means that the distance

$$\sqrt{\sum_{i=1}^{d} (o_i - s_i)^2} = \sqrt{\sum_{i=1}^{d} n_i^2}$$

between $\vec{s}$ and $\vec{o}$ is $\approx \sigma \cdot \sqrt{d}$. Let us denote this distance $\sigma \cdot \sqrt{d}$ by $\varepsilon$.

Let us, for simplicity, consider the case when $d = 2$, and when $A$ is a convex polygon. Then, we can divide all points $p$ from the exterior of $A$ that are $\varepsilon$-close to $A$ into several zones depending on what part of $A$ is the closest to $p$:

- one of the *edges* (*sides*), or
- one of the *vertices.*

Geometrically, the set of all points for which the closest point $a \in A$ belongs to the *edge* $e$ is bounded by the straight lines orthogonal (perpendicular) to $e$. The total length of this set is therefore equal to the length of this particular edge; hence, the total length of all the points that are the closest to all the sides is equal to the *perimeter* of the polygon. This total length thus does not depend on $\varepsilon$ at all.

On the other hand, the set of all the points at the distance $\varepsilon$ from $A$ grows with the increase in $\varepsilon$; its length grows approximately as the length of a circle, i.e., as const$\cdot\varepsilon$.

When $\varepsilon$ increases, the (constant) perimeter is a vanishing part of the total length. Hence, for large $\varepsilon$:

- the fraction of the points that are the closest to one of the edges tends to 0, while
- the fraction of the points $p$ for which the *closest* is one of the *vertices* tends to 1.

Similar arguments can be repeated for any dimension. For the same noise level $\sigma$, when $d$ increases, the distance $\varepsilon = \sigma \cdot \sqrt{d}$ also increases, and therefore, for large $d$, for "almost all" observed points $\vec{o}$, the reconstructed signal is one of the extreme points of the *a priori* set $A$.

**Which parameter values are more probable: case of scale-scale-invariance.** Let us show how the above general idea can be applied to the case of scale-scale-invariance, when we know that $y$ is increasing with $x$. In this case, the value $a$ can take any values from 0 to $\infty$, so the extreme cases are $a = 0$ and $a = \infty$.

Of course, literally taking $a = 0$ or $a = \infty$ makes no sense, since for each value $x+x_0$, the power $(x+x_0)^0$ is simply equal to 1 – i.e., does not depend on $x$ at all, while $(x+x_0)^\infty$ is either 0 (if $|x+x_0| < 1$) or infinity (if $|x+x_0| > 1$). So, to get non-trivial expressions, instead of directly substituting $a = 0$ or $a = \infty$ into the above formula, we need to consider limit cases when $a \to 0$ or $a \to \infty$.

Let us first consider the case $a \to 0$. In general, we have

$$(x + x_0)^a = (\exp(\ln(x + x_0)))^a = \exp(a \cdot \ln(x + x_0)).$$

For small $a \approx 0$, we can expand this expression in Taylor series and keep only linear terms in this expression:

$$(x + x_0)^a \approx 1 + a \cdot \ln(x + x_0).$$

Thus, for small $a$, the expression (25) tends to a linear transformation of a logarithm:

$$y = c_0 + c_1 \cdot \ln(x + x_0). \tag{14}$$

The case when $a \to \infty$ can be obtained from this case if we take into account that when $y$ is related to $x$ by a formula (25) with some $a$, then $x$ is related to $y$ by a similar formula, but with an exponent $1/a$. When $a \to 0$, then $1/a \to \infty$. So, the limit dependence corresponding to $a \to \infty$ is the inverse of the dependencies corresponding to $a \to 0$, i.e., a linear transformation of the exponential function:

$$y = c_0 + c_1 \cdot \exp(k \cdot x). \tag{15}$$

# 3 Need to Combine Dependencies Corresponding to Basic Invariances

**Need to combine different types of dependencies.** In some cases, we have one of the dependencies corresponding to basic invariances – e.g., the power law. However, in many other applications, empirical dependencies are more complex. How can we describe such more complex dependencies?

**First natural idea: taking intermediate quantities into account.** A natural idea is to take into account that in nature, dependencies are rarely direct: usually, when we see that a change in a quantity $x$ leads to a change in a quantity $y$, this means that:

- a change in $x$ changes some intermediate quantity $x_1$,
- the change in $x_1$, in turn, leads to the change in some other intermediate quantity $x_2$, etc.,
- until we finally teach some quantity $x_k$ that directly affects $y$.

To describe this complex dependence, we need to describe:

- how $x_1$ depends on $x$, we will denote the corresponding dependence by $x_1 = f_1(x)$,
- how $x_2$ depends on $x_1$, we will denote the corresponding dependence by $x_2 = f_2(x_1)$, etc.,
- and how $y$ depends on $x_k$, we will denote the corresponding dependence by $y = f_{k+1}(x_k)$.

Then, we have

$$y = f_{k+1}(x_k) = f_{k+1}(f_k(x_{k-1})) = \ldots =$$
$$f_{k+1}(f_k(\ldots(f_2(f_1(x)))\ldots)).$$

In other words, the function $f(x)$ describing the (indirect) dependence between $x$ and $y$ is a composition of several functions $f_1(x)$, $f_2(x_1)$, ..., $f_{k+1}(x_k)$ describing direct dependencies.

*Comment.* It should be mentioned that a combination of several dependencies does not always lead to new functions. For example, one can easily check that a composition of power laws is also a power law: indeed, e.g., if $x_1 = f_1(x) = c_1 \cdot x^{a_1}$ and $x_2 = f_2(x_1) = c_2 \cdot x_1^{a_2}$, then

$$x_2 = c_2 \cdot (c_1 \cdot x^{a_1})^{a_2} = (c_1 \cdot c_1^{a_2}) \cdot x^{a_1 \cdot a_2},$$

i.e., the dependence of $x_2$ on $x$ has the form $x_2 = c \cdot x^a$, where $c = c_2 \cdot c_1^{a_2}$ and $a = a_1 \cdot a_2$.

**Second natural idea: combining different dependencies between the same two quantities.** Sometimes, we have to combine the results of two different effects. If the effect of the first mechanism is denoted by $q_1$ and the effect of the second one by $q_2$, then a natural way to combine them is to consider some function

$$q = F(q_1, q_2). \tag{16}$$

What should be the properties of this combination function?

If one the effects is missing, then the overall effect should coincide with the other effect, so we should have $F(0, q_2) = q_2$ and $F(q_1, 0) = q_1$ for all $q_1$ and $q_2$.

If we combine two effects, it should not matter in what order we consider them, i.e., we should have

$$F(q_1, q_2) = F(q_2, q_1) \tag{17}$$

for all $q_1$ and $q_2$. In mathematical terms, the combination operation $F(q_1, q_2)$ should be *commutative*.

Similarly, if we combine three effects, the result should not depend on the order in which we combine them, i.e., that we should have

$$F(F(q_1, q_1), q_3) = F(q_1, F(q_2, q_3)) \tag{18}$$

for all $q_1$, $q_1$, and $q_3$. In mathematical terms, the combination operation $F(q_1, q_2)$ should be *associative*.

It is also reasonable to require that if we increase one of the effects, then the overall effect will increase, i.e., that the function $F(q_1, q_2)$ should be *strictly monotonic* in each of the variables: if $q_1 < q_1'$, then we should have

$$F(q_1, q_2) < F(q_1', q_2).$$

It is also reasonable to require that small changes to $q_i$ should lead to small changes in the overall effect, i.e., that the function $F(q_1, q_2)$ should be *continuous.*

Finally, it is reasonable to require that the operation $F(q_1, q_2)$ be *invariant*, in the sense that if we apply some physically meaningful transformation to the inputs $q_1$ and $q_2$ and apply the function $F(q_1', q_2')$ to the transformed values, then the result $q' = F(q_1', q_2')$ of this application should be equal to the result of applying the same transformation to $q = F(q_1, q_2)$.

**Second general idea: case of scale-invariance.** Let us first consider the case of scale-invariance. If this case, the above invariance takes the following form: if $q = F(q_1, q_2)$, then for every $\lambda > 0$,

- if we take $q_i' = \lambda \cdot q_i$ and $q' = \lambda \cdot q$,
- then we should have $q' = F(q_1', q_2')$.

It is known – see, e.g., [47] – that every commutative, associative, strictly monotonic, continuous, and scale invariant combination operation for which $F(q_1, 0) = q_1$ has the form

$$F(q_1, q_2) = (q_1^p + q_2^p)^{1/p} \tag{19}$$

for some $p > 0$.

**Second general idea: other possible cases.** In this paper, we only consider applications of this idea for the case of scale-invariance. However, it makes sense to consider more general cases of invariance, they may turn out to be helpful for other applications.

For example, to analyze what this idea will lead to for shift-invariance, we can take into account that a shift $Q \mapsto Q+Q_0$ is equivalent to scaling $q \mapsto c \cdot q$ for the auxiliary quantity $q = \exp(Q)$; here $c = \exp(Q_0)$. Thus, shift-invariance for a quantity $Q$ is equivalent to scale-invariance for the related quantity $q = \exp(Q)$. Hence:

- to describe all possible associative, commutative, associative, strictly monotonic, continuous, and *shift*-invariant combination operations on the values $Q$,
- it is sufficient to describe all possible commutative, associative, strictly monotonic, continuous, and *scale*-invariant combination operations on the quantities $q = \exp(Q)$.

To describe each such operation in terms of the original quantity $Q$, we therefore need to do the following:

- first, we transform original values $Q_1$ and $Q_2$ into the values $q_1 = \exp(Q_1)$ and $q_2 = \exp(Q_2)$;
- then, we use the formula (19) – coming from scale-invariance – to combine the values $q_1$ and $q_2$ into a new value $q = F(q_1, q_2)$, and
- finally, we transform the new value $q$ back into the original scale, by taking $Q = \ln(q)$.

As a result, we get the transformation

$$Q = \frac{1}{p} \cdot \ln(\exp(p \cdot Q_1) + \exp(p \cdot Q_1)).$$

*Comment.* It is desirable to develop similar formulas for other invariances.

## 4 Case Study: Pavement Engineering

In this section, we show that the invariance approach can, in some reasonable sense, explain empirical formulas related to all stages of pavement engineering.

**What we mean by explaining.** In many practical cases, we have empirical dependencies $y = f(x)$ or, more generally, $y = f(x_1, \ldots, x_v)$. By *explaining*, we mean formulating some physically reasonable properties that would uniquely determine the empirical dependence – either fully uniquely or modulo a small number of constants that still need to be determined empirically.

For example, if we observe a power law $y = c \cdot x^a$ – a dependence corresponding to scale-scale invariance – then a natural explanation if that this dependence is scale-scale invariant. Indeed, as we have mentioned, the property of scale-scale invariance leads to a power law, and the only thing that remains to be determined is the exact values of the parameters $c$ and $a$.

*Comment.* In this paper, we understand the word "explain" in this limited mathematical sense. Of course, from the physics viewpoint, important questions remain unexplained: e.g., why for this dependence, there is scale-scale invariance, while in other similar situations, the dependence is not described by a power law – and is, thus, not scale-scale invariant. In some of the following cases, we provide such a physical explanation as well — e.g., we explain why for some quantities, scaling is more appropriate, while for other quantities, shift is more appropriate.

We do not have such a physical explanation for all these cases. We believe, however, that our "mathematical" explanations can help to come up with such physical explanations as well.

**Stages of pavement engineering.** In order to describe how the invariance ideas can be applied on different stages of pavement engineering, let us first describe these stages.

First, we need to build the road so that the pavement will be sufficiently stiff. The upper layer of the road is designed by us, so we can make it as stiff as needed. However, for the road to be stiff, the underlying soil layer must also be sufficiently stiff. If the soil itself is not sufficiently stiff, we need to enhance its stiffness by adding additional material. In determining how much additional material we need, engineers use empirical formulas. In the first subsection of this section, we show that the above invariance ideas can lead to a theoretical explanation for these formulas.

An additional aspect is that the road must be sufficiently stiff under all possible weather conditions, including rare but possible heavy rains. Again, the top layers of the road are not much affected by the rain, but the soil can be seriously affected. To predict how stiff the road will be in different weather conditions, engineers also use empirical formulas – formulas whose theoretical explanation is provided in the second subsection of this section.

After the road is built and is being used, we need to periodically gauge its quality and, if needed, decide which segments of the road need maintenance and/or repair. To make these decisions, we need to be able to predict how the quality of the pavement will change with time: first we should repairs the segments that are expected to deteriorate quickly. In gauging the pavement quality and in predicting how this quality will change with time, engineers also use empirical formulas – formulas that we explain in the remaining subsections of this section.

## 4.1 Explaining Empirical Formulas Used in Pavement Design

**Need for lime stabilized pavement layers.** To have a stable road, it is often necessary to enhance the mechanical properties of the underlying soil layer. The most cost-efficient way of this enhancement is to mix soil with lime (sometimes coal fly ash is also added). Water is then added to this mix, and after a few days, the upper level of the soil becomes strengthened. The needed amount of lime depends on the soil.

**How to determine the optimal amount of lime.** To determine the proper amount of lime, soil specimens are brought into the lab, mixed with different amounts of lime, and tested. All chemical processes become faster when the temperature increases. So, to speed up the testing process, instead of simply waiting for several weeks as in the field, practitioners heat the sample to a higher temperature, thus speeding up the strengthening process; this higher-temperature speed-up is known as *curing*.

Based on the testing results, we need to predict the strength of the soil in the field for different possible lime amounts – and thus, select the lime amount that guarantees the desired strength.

**Need for formulas describing the dependence of strength on curing temperature and other parameters.** To be able to make this prediction, we need to know how the strength depends on the lime content $L$ (which is usually measured in percentage of lime in the dry weight of the mix). To be more precise, we need a formula with one or more parameters depending on the soil. We can then:

- determine the parameters based on the testing results, and then
- use the corresponding formula to predict the soil strength.

It turns out that the resulting empirical formulas differ depending on the porosity $\eta$ of the mix, i.e., the percentage of voids in the overall volume of the soil: for different values of $\eta$, we have, in general, different dependencies on lime content $L$.

**Known empirical formulas.** The mix is isotropic, so its mechanical strength can be characterized by two parameters:

- unconfined compressive strength $q_u$ that describes the smallest value of pressure (force over area) applied at the top of a cylindrical sample at which this sample fails;
- the tensile strength $q_t$ is when the force is applied orthogonally to the cylinder's axis.

For both types of strength $q$, the empirical formulas describing the dependence of strength on $\eta$ and $L$ are

$$q = c_1 \cdot \eta^{e_\eta} \cdot L^{e_L}, \tag{20}$$

for some parameters $c_1$, $e_\eta$, and $e_L$; see, e.g., [14, 15, 25, 49, 52].

The corresponding constant $c_1$ depends on the dry density $\rho$. The dependence on $\rho$ takes the form

$$c_1 = c_2 \cdot \rho^{e_\rho} \tag{21}$$

for some constants $c_1$ and $e_\rho$; see, e.g., [25]. Substituting the formula (2) into the formula (1), we get

$$q = c_2 \cdot \rho^{e_\rho} \cdot \eta^{e_\eta} \cdot L^{e_L}. \tag{22}$$

**Our explanation.** All three formulas are power laws – and thus, can be explained by scale-scale invariance; see [45] for details.

## 4.2 Explaining Empirical Formulas that Predict How the Pavement Will Behave in Different Weather Conditions

**Need to take into account water content in road design and management.** It is important to make sure that the roads retain sufficiently stiff under all possible weather conditions. Out of different weather conditions, the most important effect on the road stiffness is produced by rain: rainwater penetrates the reinforced-soil foundation of the pavement (called *subgrade soil*) that underlies more stiff layers of the road, and the presence of water decreases the road's stiffness.

**Towards the empirical formulas.** The mechanical effect of water can be described by the corresponding pressure $h$. In transportation engineering, this pressure is known as *suction*.

This pressure is easy to explain based on every person's experience of walking on an unpaved road:

- when the soil is dry, it exerts high pressure on our feet, thus preventing shoes from sinking, and keeping the surface of the road practically intact;
- on the other hand, when the soil is wet, the pressure drastically decreases; as a result, the shoes sink into the road, and leave deep tracks.

Similarly, the car's wheels sink into a wet road and leave deep tracks. The effect is not so prominent on paved roads, but still moisture affects the road quality.

To describe this effect in quantitative terms – and thus, to predict the effect of different levels of water saturation – we need to find the relation between the water content and the suction. Usually, for historical reasons, this effect is described as the dependence of water content $\theta$ on suction $h$ – but we can also invert this dependence and consider the dependence of suction $h$ on the water content $\theta$. The dependence of $\theta$ on $h$ is known as the *soil-water characteristic curve* (SWCC, for short).

Until the 1990s, this dependence was described by a power law $\theta = c \cdot h^{-m}$ for some parameters $c$ and $m > 0$. (Since the suction decreases with the increase in water content, the exponent $-m$ should be negative.)

This power law formula was first proposed in [10] by R. H. Brooks and A. T. Corey. Many empirical studies confirmed this dependence; see, e.g., [11, 13, 22, 23, 27, 48, 56].

This law works reasonably well for intermediate values of $\theta$. However, this formula is not perfect. For example, for $\theta \to 0$, this formula – or, to be precise, the inverse formula $h = \text{const} \cdot \theta^{-1/m}$ – implies the physically unreasonable infinite value of suction pressure. For the important case when the soil is heavily saturated with water – i.e., when $\theta$ is large – it is also not in good accordance with the empirical data.

As a result of this imperfection, in practice, until the 1990s, the results of the above power law formula were usually corrected by experts. To get a better fit with the observations and with the expert estimates, the paper [21] by D. G. Fredlung and A. Xing proposed a more complex formula

$$\theta = \text{const} \cdot \left(\ln(e + (h/a)^b)\right)^{-c}, \tag{23}$$

for some parameters $a$, $b$, and $c$. This formula has been experimentally confirmed for a wide range of values of the water content $\theta$; see, e.g., [21, 37, 57]. At present (2020), this formula – with a minor modification that we will discuss later – is recommended by the US standards; see, e.g., p. 209 of Appendix DD1 "Resilient Modulus as Function of Soil Moisture – Summary of Predictive Models" of [32] and Chapter 5, p. 42 of [30] (see also Section 2.3 of [3]).

*Comment.* In many applications, to get an even more accurate description, practitioners multiply the right-hand side of the formula (23) by an additional factor

$$C(h) = 1 - \frac{\ln\left(1 + \dfrac{h}{h_r}\right)}{\ln\left(1 + \dfrac{h_0}{h_r}\right)} \tag{24}$$

for some values $h_r$ and $h_0$.

**Towards an explanation.** As we have mentioned earlier, historically the first formulas for describing the soil-water characteristic curves were indeed the power law formulas – and the above derivation explains why these formula provide a good first approximation. However, as we also mentioned earlier, the power law is a crude approximation, we need to go beyond power laws. Thus, we need to consider the case when we have several sequential transformations.

In all these transformations are scale-scale-invariant, then they are all power laws. In this case, the resulting dependence of $y$ on $x$ is still a power law. To get beyond the power laws, we need to take into account that for some intermediate dependencies, we may get different starting points. If we change a starting point for measuring $x$ to a different starting point which is $x_0$ units smaller the original one, then this value $x_0$ will be added to all numerical values of this quantity:

$x \mapsto x' = x + x_0$, so that $x = x' - x_0$. Similarly, if we change a starting point for measuring $y$ to a different starting point which is $y_0$ units smaller than the original one, then this value $y_0$ will be added to all numerical values of this quantity: $y \mapsto y' + y_0$, so that $y = y' - y_0$.

If in the new units $x'$ and $y'$, we have a power law dependence $y' = c \cdot (x')^a$ (motivated by scale-scale-invariance), then in the original units $x$ and $y$, we will have

$$y = y' - y_0 = c \cdot (x')^a - y_0 = c \cdot (x + x_0)^a - y_0,$$

i.e., the form

$$y = c \cdot (x + x_0)^a - y_0. \tag{25}$$

It is thus reasonable to replace one of the intermediate power-law dependencies with this more general formula.

We have argued that the most probable cases are extreme cases $a \to 0$ or $a \to \infty$ that are described by formulas (14) and (15). What will then be the resulting dependence between $x$ and $y$?

Let us start with considering the case when the intermediate transformation is described by a logarithm formula (14). In this case,

- first, we have several power-law transformations, which, as we have learned, are equivalent to a single power-law transformation; as a result, the original value $x$ is transformed into a new value $x_1 = a_1 \cdot x^{b_1}$ for some $a_1$ and $b_1$;
- then, to the resulting value $x_1$, we apply the logarithm transformation (25), resulting in

$$x_2 = c_0 + c_1 \cdot \ln(x_1 + x_0) = c_0 + c_1 \cdot \ln(a_1 \cdot x^{b_1} + x_0);$$

- finally, we again have several power-law transformations, which are equivalent to a single power-law transformation $y = a_3 \cdot x_2^{b_3}$ for some values $a_3$ and $b_3$, resulting in

$$y = a_3 \cdot (c_0 + c_1 \cdot \ln(a_1 \cdot x^{b_1} + x_0))^{b_3}. \tag{26}$$

**Let us simplify this formula.** Let us simplify this formula, to make it closer to the desired formula (23). First, we can represent $a_1 \cdot x^{b_1} + x_0$ as $c_2 \cdot (a'_1 \cdot x^{b_1} + e)$, where we denoted $c_1 \stackrel{\text{def}}{=} \dfrac{x_0}{e}$ and $a'_1 \stackrel{\text{def}}{=} \dfrac{a_1}{c_2} = \dfrac{a_1 \cdot e}{x_0}$. Then,

$$\ln(a_1 \cdot x^{b_1} + x_0) = \ln(c_2 \cdot (a'_1 \cdot x^{b_1} + e)) = \ln(c_2) + \ln(e + a'_1 \cdot x^{b_1}),$$

and thus,

$$c_0 + c_1 \cdot \ln(a_1 \cdot x^{b_1} + x_0) = c_0 + c_1 \cdot \ln(c_2) + c_1 \cdot \ln(e + a'_1 \cdot x^{b_1}),$$

i.e.,

$$c_0 + c_1 \cdot \ln(a_1 \cdot x^{b_1} + x_0) = c'_0 + c_1 \cdot \ln(e + a'_1 \cdot x^{b_1}),$$

where we denoted $c_0' \stackrel{\text{def}}{=} c_0 + \ln(c_2)$. This expression, in turn, can be described as

$$c_0 + c_1 \cdot \ln(a_1 \cdot x^{b_1} + x_0) = c_0' + c_1 \cdot \ln(e + a_1' \cdot x^{b_1}) =$$
$$c_1 \cdot (\ln(e + a_1' \cdot x^{b_1}) + c_0''),$$

where $c_0'' \stackrel{\text{def}}{=} c_0'/c_1$. Thus,

$$(c_0 + c_1 \cdot \ln(a_1 \cdot x^{b_1} + x_0))^{b_3} = c_1^{b_3} \cdot (\ln(e + a_1' \cdot x^{b_1}) + c_0'')^{b_3}.$$

Multiplying both sides by $a_3$, we conclude that the formula (26) can be described in the following form

$$y = a_3' \cdot (\ln(e + a_1' \cdot x^{b_1}) + c_0'')^{b_3}, \tag{27}$$

where $a_3' \stackrel{\text{def}}{=} a_3 \cdot c_1^{b_3}$.

**This is (almost) exactly what we want.** The empirical formula (23) can be viewed as a particular case of the above formula (27), with $c_0'' = 0$, $a_3' = \text{const}$, $a_1 = a^{-b}$, and $b_3 = -c$.

Vice versa, any expression (27) with $c_0'' = 0$ has the form (23). So, we (almost) have what we want: a theoretically justified formula: the only difference is that our formula has one more parameter $c_0''$. Who knows, maybe empirically, we can find some non-zero value of this parameter for which this formula will be even more accurate than the original empirical formula (23)?

*Comments.*

- When we describe limit cases of scale-scale-invariance, we had a choice:
  - we could have a logarithmic dependence, or
  - we could have the inverse (exponential) dependence.

  Which dependence we choose depends on which of two quantities we consider as input and which as output. If instead of the dependence $\theta(h)$, we will consider the inverse dependence $h(\theta)$, then we will get exponential function instead of the logarithmic one. Which of the two dependencies $\theta(h)$ or $h(\theta)$ is logarithmic and which is exponential cannot be determined purely theoretically, since we assume the same scale-scale-invariance property for both quantities; this must be determined empirically. In this particular case, the dependence $\theta(h)$ is logarithmic.
- The additional factor (24) can also be explained the same way: as one can see, it is exactly one of the two limit cases of power law dependency: namely, the logarithmic limit case (14).
- A detailed description of this case can be found in [44].

## 4.3 Explaining Empirical Formulas for Gauging Pavement Quality

**Estimating pavement roughness.** Estimating road roughness is an important problem. Indeed, road pavements need to be maintained and repaired. Both maintenance and repair are expensive. So, it is desirable to estimate the pavement roughness as accurately as possible:

- if we overestimate the road roughness, we will waste money and other resources on road segments which are in reasonably good shape, at the expense of other segments which may need maintenance or repair;
- if we underestimate the road roughness, the road segment will be left unrepaired and deteriorate even more – as a result of which the cost of its future repair will skyrocket.

The standard way to measure the pavement roughness is to use the International Roughness Index (IRI); see, e.g., [5, 17, 18, 50]. This measure of roughness is recommended by the US standards [5, 17, 18].

Crudely speaking, IRI describes the effect of the pavement roughness on a standardized model of a vehicle. Measuring IRI is not easy, because the real vehicles differ from this standardized model. As a result, we measure roughness by some instruments and use these measurements to estimate IRI. For example, we can:

- perform measurements by driving an available vehicle along this road segment,
- extract the local roughness characteristics from the effect of the pavement on this vehicle, and then
- use these extracted characteristics to estimate the effect of the same pavement on the standardized vehicle.

In view of this difficulty, in many cases, practitioners rely on expert estimates of the pavement roughness. The corresponding measure – estimated on a scale from 0 to 5 – is known as the Present Serviceability Rating (PSR); see, e.g., [4, 19].

**Empirical relation between measurement results and expert estimates, on the example of pavement roughness.** The empirical relation between PSR and IRI is described by the following formula:

$$\text{PSR} = 5 \cdot \exp(-0.0041 \cdot \text{IRI}). \tag{28}$$

This formula was first proposed by B. Al-Omari and M. Darter in [2], and it still remains actively used in pavement engineering; see, e.g., [7, 19, 36, 38].

**Our explanation.** This formula can be explained by shift-scale invariance; see [41] for details.

### 4.4 Predicting Pavement Quality: How Pavement Condition Index Will Change with Time

**Empirical formula.** The quality of a road pavement is described by a Pavement Condition Index (PCI) that takes into account all possible pavement imperfections [6]. The perfect condition of the road corresponds to PCI = 100, and the worst possible condition corresponds to PCI = 0.

As the pavement ages, its quality deteriorates. To predict this deterioration, practitioners use an empirical formula developed in [28]:

$$\mathrm{PCI} = 100 - \frac{R}{(\ln(\alpha) - \ln(t))^{1/\beta}}, \tag{29}$$

where $t$ is the pavement's age, and $R$, $\alpha$ are corresponding parameters.

**PCI and age: scale-invariance or shift-invariance?** We are analyzing how PCI depends on the pavement's age $t$. To apply invariances to our dependence, we need first to analyze which invariances are reasonable for the corresponding variables – PCI and age.

For age, the answer is straightforward: there is a clear starting point for measuring age, namely, the moment when the road was built. On the other hand, there is no fixed measuring unit: we can measure age in years or in months or – for good roads – in decades. Thus, for age:

- shift-invariance – corresponding to the possibility of changing the starting point – makes no physical sense, while
- scale-invariance – corresponding to the possibility of changing the measuring unit – makes perfect sense.

For PCI, the situation is similar. Namely, there is a very clear starting point – the point corresponding to the newly built practically perfect road, when PCI = 100. From this viewpoint, for PCI, shifts do not make much physical sense. If we select 100 as the starting point (i.e., as 0), then instead of the original numerical values PCI, we get shifted values PCI − 100.

A minor problem with these shifted values is that they are all negative, while it is more convenient to use positive numbers. Thus, we change the sign and consider the difference 100 − PCI.

On the other hand, the selection of point PCI = 0 is rather subjective. What is marked as PCI = 0 in a developed country that can afford to invest money into road repairs may be a passable road in a poor country, where most of the roads are, from the viewpoint of US standards, very bad; see, e.g., [24]. So, for PCI (or, to be more precise, for 100 − PCI), it probably makes sense to use scaling.

**We cannot directly apply the invariance ideas.** In view of the above analysis, we should be looking for a dependence of $y = 100 - \mathrm{PCI}$ on $x = t$ which is invariant with respect to $x$-scaling and $y$-scaling. As we have discussed in the previous section, this requirement leads to $y = A \cdot x^b$, i.e., to $100 - \mathrm{PCI} = A \cdot t^b$ and $\mathrm{PCI} = 100 - A \cdot t^b$.

This formula may be reasonable from the purely mathematical viewpoint, but in practice, it is a very crude description of what we actually observe. Thus, the direct application of invariance ideas does not lead to good results.

**Let us now apply invariance ideas indirectly.** Since we cannot apply the invariance requirements *directly* – to describe the dependence of $y = 100 - \mathrm{PCI}$ on $x = t$, a natural idea is to apply these requirements *indirectly*. Namely, we assume that there is some auxiliary intermediate variable $z$ such that:

- $y$ depends on $z$,
- $z$ depends on $x$, and
- both these $y$-on-$z$ and $z$-on-$x$ dependencies are, in some reasonable sense, invariant.

**Options.** We know that for $x$ and for $y$, only scaling makes sense. However, for the auxiliary variable $z$, in principle, both shifts and scalings may be physically reasonable. Depending on which of the two types of transformations we use for $z$ when describing $y$-on-$z$ and $z$-on-$x$ dependencies, we get four possible options:

- for both $y$-on-$z$ and $z$-on-$x$ dependencies, we use $z$-shift;
- for both $y$-on-$z$ and $z$-on-$x$ dependencies, we use $z$-scaling;
- for $y$-on-$z$ dependence, we use $z$-shift, while for $z$-on-$x$ dependence, we use $z$-scaling;
- for $y$-on-$z$ dependence, we use $z$-scaling, while for $z$-on-$x$ dependence, we use $z$-shift.

Let us consider these four cases one by one.

**Case when for both $y$-on-$z$ and $z$-on-$x$ dependencies, we use $z$-shift.** In this case, in accordance to the results presented in Section 2, we have $z = A + b \cdot (x)$ and $y = A_1 \cdot (b_1 \cdot z)$. Substituting the expression for $z$ into the formula for $y$, we get

$$y = A_1 \cdot \exp(A + b \cdot \ln(x)) = (A_1 \cdot \exp(A)) \cdot (\exp(\ln(x))^b = A_2 \cdot x^b,$$

where $A_2 \stackrel{\text{def}}{=} A_1 \cdot \exp(A)$. This is exactly the formula coming from the direct application of invariance requirements, and we already know that this formula is not very adequate for describing the experimental data.

**Case when for both $y$-on-$z$ and $z$-on-$x$ dependencies, we use $z$-scaling.** In this case, we have $z = A \cdot z^b$ and $y = A_1 \cdot z^{b_1}$. Thus, here,

$$z = A_1 \cdot \left(A \cdot x^b\right)^{b_1} = A_2 \cdot x^{b_2},$$

where $A_2 \stackrel{\text{def}}{=} A_1 \cdot A^{\alpha_1}$ and $b_2 \stackrel{\text{def}}{=} b \cdot b_1$. Thus, in this case, we also get the same formula as for the direct application of invariance.

**Case when for $y$-on-$z$ dependence, we use $z$-shift, while for $z$-on-$x$ dependence, we use $z$-scaling.** Here, $z = A \cdot x^b$ and $y = A_1 \cdot \exp(b_1 \cdot y)$, thus $y = A_1 \cdot \exp\left(b_1 \cdot A \cdot x^b\right)$, i.e., $y = A_1 \cdot \exp\left(b_2 \cdot x^b\right)$, where $b_2 \stackrel{\text{def}}{=} b_1 \cdot A$. So, for $\text{PCI} = 100 - y$ and $x = t$, we get the dependence

$$\text{PCI} = 100 - A_1 \cdot \exp\left(b_2 \cdot t^b\right). \tag{30}$$

Interestingly, this is one of the formula that was tested in [28] and which turned out to work not so well as the formula that was selected.

**Case when for $y$-on-$z$ dependence, we use $z$-scaling, while for $z$-on-$x$ dependence, we use $z$-shift.** In this case, $z = A + b \cdot \ln(x)$ and $y = A_1 \cdot z^b$, thus $y = A_1 \cdot (A + b \cdot \ln(x))^{b_1}$. So, for $\text{PCI} = 100 - y$ and $x = t$, we get

$$\text{PCI} = 100 - A_1 \cdot (A + b \cdot \ln(x))^{b_1}. \tag{31}$$

Let us show that this is indeed the desired formula (29).

Indeed, here,

$$A + b \cdot \ln(x) = (-b) \cdot \left(\left(-\frac{A}{b}\right) - \ln(x)\right). \tag{32}$$

For $\alpha \stackrel{\text{def}}{=} \exp\left(-\dfrac{A}{b}\right)$, we have $\ln(\alpha) = -\dfrac{A}{b}$, so the formula (32) takes the form $A + b \cdot \ln(x) = (-b) \cdot (\ln(\alpha) - \ln(t))$. Thus, the formula (11) takes the form

$$\text{PCI} = 100 - A_1 \cdot (-b)^{b_1} \cdot (\ln(\alpha) - \ln(t))^{b_1},$$

i.e., the desired form (1) with $R = A_1 \cdot (-b)^{b_1}$ and $\beta = -\dfrac{1}{b_1}$.

**Conclusion.** We indeed derived the empirical formula (29) for the decrease of PCI over time from the general invariance requirements. To be more precise, from the invariance requirements, we can derive two possible formulas:

- the desired formula (29) – which is in good accordance with the empirical data, and
- the alternative formula (30) – which is *not* a good fit for empirical data.

*Comment.* A detailed description of this case is given in [40, 46].

## 4.5 Predicting Pavement Quality: How the Amount of Cracks and Potholes Will Grow with Time

**Cracks and potholes.** Not only we want to predict how the pavement quality will change (deteriorate) with time, we also want to predict what exactly will deteriorate. One of the main ways pavement deteriorates is that cracks and potholes appear and start growing.

**How transportation engineers usually gauge the amount of cracks and potholes.** The amount of cracks is usually gauged the overall length $C$ of the longitudinal cracks outside the direct wheel path. The amount of potholes is usually gauged by the total area $P$ of potholes.

As the road is used, the quality of the pavement deteriorates, and the values $C$ and $P$ grow. This growth starts at some small values corresponding to the newly built road – age $t = 0$ – and continues growing until they reach the maximum – the undesirable bad state when the whole road is covered by cracks and potholes.

**Empirical formulas.** According to [31], both growths are described by similar formulas

$$C = a_C \cdot \exp(-b_C \cdot \exp(-c_C \cdot t)); \tag{33}$$

$$P = a_P \cdot \exp(-b_P \cdot \exp(-c_P \cdot t)). \tag{34}$$

**What we want: a brief reminder.** We want to find the dependence of the quantity $q$ (crack or pothole amount) on time $t$. We know:

- that the for $t = 0$, the value $q(t)$ is small positive,
- that the value $q(t)$ increases with time, and
- that the value $q(t)$ tends to some large constant value when $t$ increases.

**What are possible invariances here?** For crack amount $C$ and for pothole amount $P$, there is an absolute starting point – 0, when we have no cracks and no potholes. However, it makes sense to use different units of length and different units of area, so scaling makes perfect sense.

For time, as we have mentioned, both shift and scaling make sense.

**First idea.** If view of the above analysis, let us see if any of the above invariant dependencies satisfy the desired property.

Since for $q$, only scaling makes sense, we can only consider two possibilities: sc-sc and sh-sc. Let us consider them one by one.

**First idea: sc-sc case.** In the sc-sc case, we have $q(t) = A \cdot t^a$. Since we want a non-negative value, we have to take $A > 0$. Since we want $q(t)$ to be increasing with time, we have to take $a > 0$. However, in this case:

- $q(0)$ is zero – while we want it to be positive, and
- $q(t)$ tends to infinity as $t$ increases – while we want it to tend to some constant.

**First idea: sh-sc case.** In the sh-sc case, we have $q(t) = A \cdot \exp(a \cdot t)$. Again, since we want a non-negative value, we have to take $A > 0$. Since we want $q(t)$ to be increasing with time, we have to take $a > 0$. In this case:

- $q(0)$ is positive, which is exactly what we wanted, but
- $q(t)$ tends to infinity as $t$ increases – while we want it to tend to some constant.

**So what do we do?** The first idea does not work, so what should we do?

The above arguments about possible dependencies deal with the case when the quantity $y$ directly depend on the time $t$. However, in our case, cracks and potholes do not directly depend on time: what changes with time is stress, which, in its turn, causes the pavement to crack. In other words, instead of the direct dependence of the quantity $q$ on time:

- we have $q$ depending on some auxiliary quantity $z$, and
- we have $z$ depending on time $t$.

For both dependencies $q(z)$ and $z(t)$ we can have invariance-motivated formulas. Let us see which combinations of these formulas provide the desired properties of the resulting dependence $q(t) = q(z(t))$ – that this value is positive for $t = 0$, increases for $t > 0$, and tends to a finite limit when $t \to \infty$.

**Possible options of the $q(z)$ dependence.** Since for $q$, only scaling is possible, for possible dependencies $q(z)$, we have either the sc-sc option $q(z) = A \cdot z^a$ or the sh-sc option $q(z) = A \cdot \exp(a \cdot z)$.

**First option $q(z) = A \cdot z^a$.** In this option, when $q(z)$ is sc-sc, it does not make sense to consider sh-sc or sc-sc options for $z(t)$, since, as one can check, this will be equivalent to sh-sc or sc-sc invariance for $q(t)$, and we have already shown that this is not possible. So, to go beyond previously considered options, we need to consider two remaining options for $z(t)$: sh-sh option $z(t) = a_1 \cdot t + C_1$, and sc-sh option $z(t) = a_1 \cdot \ln(t) + C_1$.

In the first case, we have $q(t) = A \cdot z^a = A \cdot (a_1 \cdot t + C_1)^a$. We can equivalently describe it as $q(t) = A_1 \cdot (t + c_2)^a$, where $A_1 = A \cdot (a_1)^a$ and $c_2 = \dfrac{C_1}{a_1}$. The need to have positive values of $q$ implies $A > 0$, the need to have $q(t)$ increasing leads to $a > 0$, but then, for $t \to \infty$, the resulting expression tends to infinity – while we want it bounded.

In the second case, we have $q(t) = A \cdot z^a = A \cdot (a_1 \cdot \ln(t) + C_1)^a$. Similarly to the first case, we can equivalently describe this expression as $q(t) = A_1 \cdot (\ln(t) + c_2)^a$, with $A_1 = A \cdot (a_1)^a$ and $c_2 = \dfrac{C_1}{a_1}$. The need to have positive values of $q$ implies $A > 0$, the need to have $q(t)$ increasing leads to $a > 0$, but then, for $t \to \infty$, the resulting expression also tends to infinity – while we want it bounded.

**Second option** $q(z) = A \cdot \exp(a \cdot z)$**.** In this option, when $q(z)$ is sh-sc, it does not make sense to consider sh-sh or sc-sh options for $z(t)$, since, as one can check, this will be equivalent to sh-sc or sc-sc invariance for $q(t)$, and we have already shown that this is not possible. So, to go beyond previously considered options, we need to consider two remaining options for $z(t)$: sc-sc option $z(t) = A_1 \cdot t^{a_1}$, and sh-sc option $z(t) = A_1 \cdot \exp(a_1 \cdot t)$.

In the first case, $q(t) = A \cdot \exp(a \cdot z) = A \cdot \exp((a \cdot A_1) \cdot t^{a_1})$. The need to have positive values of $q$ implies $A > 0$. The behavior of this expression depends on the sign of the product $a \cdot A_1$.

- If $a \cdot A_1 > 0$, then the need to have $q(t)$ increasing leads to $a_1 > 0$, but then, for $t \to \infty$, the resulting expression tends to infinity – and we want it bounded.
- If $a \cdot A_1 < 0$, then the need to have $q(t)$ increasing leads to $a_1 < 0$, but then, for $t \to 0$, we have $t^{-|a_1|} \to \infty$, hence $(a \cdot A_1) \cdot t^{-|a_1|} \to -\infty$, and $q(t) = A \cdot \exp((a \cdot A_1) \cdot t^{-|a_1|}) \to 0$, but we want the value $q(0)$ to be positive.

So, the only possible case is the second case, when

$$q(t) = A \cdot \exp(a \cdot z) = A \cdot ((a \cdot A_1) \cdot \exp(a_1 \cdot t)),$$

which is exactly the desired formulas (33) and (34).

**Conclusion.** So, we can conclude that the only invariance-motivated dependence $q(t)$ for which $q(0) > 0$ and $q(t)$ increases until some finite number is the dependence (33) and (34). Thus, we have indeed justified the empirical dependencies (33) and (34).

*Comment.* A detailed description of this case is given in [42].

### 4.6 Predicting Pavement Quality: How Crack Size Will Change with Time

**Which cracks should be repaired first?** Under stress, cracks appear in constructions. They appear in buildings, they appear in brides, they appear in pavements, they appear in engines, etc. Once a crack appears, it starts growing.

Cracks are potentially dangerous. Cracks in an engine can lead to a catastrophe, cracks in a pavement makes a road more dangerous and prone to accidents, etc. It is therefore desirable to repair the cracks.

In the ideal world, each crack should be repaired as soon as it is noticed. This is indeed done in critical situations – e.g., after each flight, the Space Shuttle was thoroughly studied and all cracks were repaired.

However, in most other (less critical) situations, for example, in pavement engineering, our resources are limited. In such situations, we need to decide which cracks to repair first. A natural idea is to concentrate our efforts on cracks that, if unrepaired, will become most dangerous in the future. For that, we need to be able to predict how each crack will grow, e.g., in the next year.

Once we are able to predict how the current cracks will grow, we will be able to concentrate our limited repair resources on most potentially harmful cracks.

**How cracks grow: a general description.** In most cases, stress comes in cycles: the engine clearly goes through the cycles, the road segment gets stressed when a vehicle passes through it, etc. Thus, the crack growth is usually expressed by describing how the length $a$ of the pavement changes during a stress cycle at which the stress is equal to some value $\sigma$. The increase in length is usually denoted by $\Delta a$. So, to describe how a crack grows, we need to find out how $\Delta a$ depends on $a$ and $\sigma$:

$$\Delta a = f(a, \sigma), \tag{35}$$

for some function $f(a, \sigma)$.

**Case of very short cracks.** The first empirical formula – known as Wöhler law – was proposed to describe how cracks appear. In the beginning, the length $a$ is 0 (or very small), so the dependence on $a$ can be ignored, and we have

$$\Delta a = f(\sigma), \tag{36}$$

for some function $f(\sigma)$. Empirical data shows that this dependence is a power law, i.e., that

$$\Delta a = C_0 \cdot \sigma^{m_0}, \tag{37}$$

for some constants $C_0$ and $m_0$.

**Practical case of reasonable size cracks: Paris law.** Very small cracks are extremely important in critical situations: since there, the goal is to prevent the cracks from growing. In most other practical situations, small cracks are usually allowed to grow, so the question is how cracks of reasonable size grow.

Several empirical formulas have been proposed. In 1963, P. C. Paris and F. Erdogan compared all these formulas with empirical data, and came up with a new empirical formula that best fits the data:

$$\Delta a = C \cdot \sigma^m \cdot a^{m'}. \tag{38}$$

This formula – known as *Paris Law* or *Paris-Erdogan Law* – is still in use; see, e.g., [9, 26].

**Usual case of Paris law.** Usually, we have $m' = m/2$, in which case the formula (38) takes the form

$$\Delta a = C \cdot \sigma^m \cdot a^{m/2} = C \cdot (\sigma \cdot \sqrt{a}\,)^m. \tag{39}$$

The formula (38) is empirical, but the dependence $m' = m/2$ has theoretical explanations. One of such explanations is that the stress acts randomly at different parts of the crack. According to statistics, the standard deviation $s$ of the sum of $n$ independent variables each of which has standard deviation $s_0$ is equal to $s = s_0 \cdot \sqrt{n}$; see, e.g., [51]. So, on average, the effect of $n$ independent factors is proportional to $\sqrt{n}$. Thus, for a crack of length $a$, consisting of $a/\delta_a$ independent parts, the overall effect $K$ of the stress $\sigma$ is proportional to

$$K = \sigma \cdot \sqrt{n} \sim \sigma \cdot \sqrt{a}. \tag{40}$$

This quantity $K$ is known as *stress intensity.* For the power law

$$\Delta a = C \cdot K^m, \tag{41}$$

this indeed leads to

$$\Delta a = \text{const} \cdot (\sigma \cdot \sqrt{a}\,)^m = \text{const} \cdot \sigma^m \cdot a^{m/2}, \tag{42}$$

i.e., to $m' = m/2$.

**Empirical dependence between $C$ and $m$.** In principle, we can have all possible combinations of $C$ and $m$. Empirically, however, there is a relation between $C$ and $m$:

$$C = c_0 \cdot b_0^m; \tag{43}$$

see, e.g., [12, 16] and references therein.

**Beyond Paris law.** As we have mentioned, Paris law is only valid for reasonably large crack lengths $a$. It cannot be valid for $a = 0$, since for $a = 0$, it implies that $\Delta a = 0$ and thus, that cracks cannot appear by themselves – but they do. To describe the dependence (35) for all possible values $a$, the paper [8] proposed to use the expression (38) with different values of $C$, $m$, and $m'$ for different ranges of $a$. This worked OK, but not perfectly.

The best empirical fit came from the generalization of Paris law proposed in [39]:

$$\Delta a = C \cdot \sigma^m \cdot \left(a^\alpha + c \cdot \sigma^\beta\right)^\gamma. \tag{44}$$

Empirically, we have $\alpha \approx 1$.

**How can we use scale invariance here?** It would be nice to apply scale invariance to crack growth. However, we cannot directly use it: indeed, in the above arguments, we assumed that $y$ and $x_i$ are different quantities, measured by different units, but in our case $\Delta a$ and $a$ are both lengths. What can we do?

To apply scale invariance, we can recall that in all applications, stress is periodic: for an engine, we know how many cycles per minute we have, and for a road, we also know, on average, how many cars pass through the give road segment. In both cases, what we are really interested in is how much the crack will grow during some time interval – e.g., whether the road segment needs repairs right now or it can wait until the next year. Thus, what we are really interested in is not the value $\Delta a$, but the value $\dfrac{da}{dt}$ which can be obtained by multiplying $\Delta a$ and the number of cycles per selected time unit.

Since the quantities $\dfrac{da}{dt}$ and $\Delta a$ differ by a multiplicative constant, they follow the same laws as $\Delta a$ – but for $\dfrac{da}{dt}$, we already have different measuring units and thus, we can apply scale invariance.

**So, let us apply scale invariance.** For the case of one variable, scale invariance leads to the formula (3.20), which explains Wöhler law.

For the case of several variables we similarly get the formula (8), which explains Paris law (38).

Thus, both Wöhler and Paris laws can indeed be theoretically explained – by scale invariance.

**Scale-scale-invariance also explains how $C$ depends on $m$: idea.** Let us show that scale-scale-invariance can also the explain the dependence (43) between the parameters $C$ and $m$ of the Paris law (38).

Indeed, the fact that the coefficients $C$ and $m$ describing the Paris law are different for different materials means that, to determine how a specific crack will grow, it is not sufficient to know its stress intensity $K$, there must be some other characteristic $z$ on which $\Delta a$ depends:

$$\Delta a = f(K, z). \tag{45}$$

**Let us apply scale invariance.** If we apply scale invariance to the dependence of $\Delta a$ on $K$, then we can conclude that this dependence is described by a power law, i.e., that

$$\Delta a(K, z) = C(z) \cdot K^{m(z)}, \tag{46}$$

where, in general, the coefficients $C(z)$ and $m(z)$ may depend on $z$. It is well known that if we go to log-log scale, i.e., consider the dependence of $\ln(\Delta a)$ on $\ln(K)$, then the dependence becomes linear. Indeed, if we take logarithms of both sides of the equality (17), we conclude that

$$\ln(\Delta a(K, z)) = m(z) \cdot \ln(K) + \ln(C(z)). \tag{47}$$

Similarly, if we apply scale invariance to the dependence of $\Delta a$ on $z$, we also get a power law

$$\Delta a(K, z) = C'(K) \cdot z^{m'(K)} \tag{48}$$

for some values $C'(K)$ and $m'(K)$, i.e., in log-log scale,

$$\ln(\Delta a(K, z)) = m'(K) \cdot \ln(z) + \ln(C'(k)). \tag{49}$$

The logarithm $\ln(\Delta a(K, z))$ in linear in $\ln(K)$ and linear in $\ln(z)$, thus it is a bilinear function of $\ln(K)$ and $\ln(z)$. A general bilinear function has the form:

$$\ln(\Delta a(K, z)) = a_0 + a_K \cdot \ln(K) + a_z \cdot \ln(z) + a_{Kz} \cdot \ln(K) \cdot \ln(z), \tag{50}$$

i.e., the form

$$\ln(\Delta a(K,z)) = (a_0 + a_z \cdot \ln(z)) + (a_K + a_{Kz} \cdot \ln(z)) \cdot \ln(K). \tag{51}$$

By applying $\exp(t)$ to both sides of the formula (51), we conclude that the dependence of $\Delta a$ on $K$ has the form

$$\Delta a = C \cdot K^m, \tag{52}$$

where

$$C = \exp(a_0 + a_z \cdot \ln(z)) \tag{53}$$

and

$$m = a_K + a_{Kz} \cdot \ln(z). \tag{54}$$

From (54), we conclude that $\ln(z)$ is a linear function of $m$, namely, that

$$\ln(z) = \frac{1}{a_{Kz}} \cdot m - \frac{a_K}{a_{Kz}}. \tag{55}$$

Substituting this expression for $\ln(z)$ into the formula (53), we can conclude that

$$C = \exp\left(\left(a_0 - \frac{a_K \cdot a_z}{a_{Kz}}\right) + \frac{a_z}{a_{Kz}} \cdot m\right), \tag{56}$$

i.e., the desired formula (43), $C = c_0 \cdot b_0^m$, with

$$c_0 = \exp\left(a_0 - \frac{a_K \cdot a_z}{a_{Kz}}\right) \tag{57}$$

and

$$b_0 = \exp\left(\frac{a_z}{a_{Kz}}\right). \tag{58}$$

Thus, the empirical dependence (43) of $C$ on $m$ can also be explained by scale invariance.

**Generalized Paris Law: analysis of the problem.** Let us show that scale invariance can also explain the generalized Paris law (44).

So far, we have justified two laws: Wöhler law (37) that describes how cracks appear and start growing, and Paris law (38) that describes how they grow once they reach a certain size. In effect, these two laws describe two different mechanisms for crack growth. To describe the joint effect of these two mechanisms, we need to combine the effects of both mechanisms.

**This explains the generalized Paris law.** Scale-invariance requires that the combination has the form (19). If we substitute the expression (37) instead of $q_1$ and the expression (38) instead of $q_2$ into the formula (19), we get

$$\Delta a = \left((C_0 \cdot \sigma^{m_0})^p + \left(C \cdot \sigma^m \cdot a^{m'}\right)^p\right)^{1/p} =$$

$$\left(C_0^p \cdot \sigma^{m_0 \cdot p} + C^p \cdot \sigma^{m \cdot p} \cdot a^{m' \cdot p}\right)^{1/p} =$$

$$C \cdot \sigma^m \cdot \left(a^{m' \cdot p} + \left(\frac{C_0}{C}\right)^p \cdot \sigma^{(m-m_0) \cdot p}\right)^{1/p}, \tag{59}$$

i.e., we get the desired formula (44), with $\alpha = m' \cdot p$, $c = \left(\frac{C_0}{C}\right)^p$, $\beta = (m - m_0) \cdot p$, and $\gamma = 1/p$.

Thus, the generalized Paris law can also be explained by scale invariance.

*Comment.* A detailed description of this case is given in [43].


## Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes).

The authors are greatly thankful to Radim Belohlavek for his encouragement, and to the anonymous referees for valuable suggestions.


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# A Appendix: Which Dependencies Are Invariant: Derivations

**Which dependencies are scale-scale-invariant: proof for the smooth case.** In the main text, we showed that scale-scale-invariance leads to the functional equation (1). Let us show that all its smooth solutions $f(x)$ are decsribed by the power law.

Indeed, from the formula (1), we can conclude that the function $\mu(\lambda)$ is equal to the ratio of two differentiable functions $\mu(\lambda) = \dfrac{f(\lambda \cdot x)}{f(x)}$ and is, thus, differentiable too.

Since both functions $f(x)$ and $\mu(\lambda)$ are differentiable, we can differentiate both sides of the formula (1) with respect to $\lambda$. After plugging in $\lambda = 1$, we get

$$x \cdot \frac{df}{dx} = a \cdot f,$$

where we denoted

$$a \stackrel{\text{def}}{=} \frac{d\mu}{d\lambda}_{|\lambda=1}.$$

We can separate the variables in this formula if we divide both sides by $f$ and by $x$, then we get:

$$\frac{df}{f} = a \cdot \frac{dx}{x}.$$

Integrating both sides, we get $\ln(f) = a \cdot \ln(x) + C$, where $C$ is the integration constant. Applying the exponential function to both sides of this formula, we get $f = c \cdot x^a$, where we denoted $c \stackrel{\text{def}}{=} \exp(C)$. So, every scale-scale-invariant dependence is a power law.

**Which dependencies are shift-shift-invariant?** In the main text, we have shown that shift-shift invariance leads to the functional equation (9). Let us show that the only smooth solutions of this functional equation are linear functions.

Indeed, from the formula (9), we can conclude that the function $y_0(x_0)$ is equal to the difference of two differentiable functions $y_0(x_0) = f(x + x_0) - f(x)$ and is, thus, differentiable too.

Since both functions $f(x)$ and $y_0(x_0)$ are differentiable, we can differentiate both sides of the formula (9) with respect to $x_0$. After plugging in $x_0 = 0$, we get

$$\frac{df}{dx} = a,$$

where we denoted

$$a \stackrel{\text{def}}{=} \frac{dy_0}{dx_0}_{|x_0=0}.$$

We can separate the variables in this formula if we multiply both sides by $dx$, then we get:

$$df = a \cdot dx.$$

Integrating both sides, we get $f = a \cdot x + C$, where $C$ is the integration constant. So, every shift-shift-invariant dependence is a linear function.

**Shift-to-scaling invariance.** Let us show that the only smooth solutions of the functional equation (10) are exponential functions.

Indeed, since the function $f(x)$ is differentiable, then the function $\mu(x_0) = \dfrac{f(x + x_0)}{f(x)}$ is differentiable too. Thus, we can differentiate both sides of the equation (10) with respect to $x_0$. As a result, we get

$$f'(x + x_0) = \mu'(x_0) \cdot f(x). \tag{A1}$$

In particular, for $x_0 = 0$, we get

$$\frac{df}{dx} = a \cdot f, \tag{A2}$$

where $a \stackrel{\text{def}}{=} \mu'(0)$. We can separate the variables $x$ and $f$ if we multiply both sides of the equality (A2) by $\dfrac{dx}{f}$, then we get

$$\frac{df}{f} = a \cdot dx. \tag{A3}$$

Integrating both sides, we get

$$\ln(f) = a \cdot x + C, \tag{A4}$$

where $C$ is the integration constant. Applying the function $\exp(z)$ of both sides of the equality (A4), we get the desired expression $f(x) = A \cdot \exp(a \cdot x)$, with $A = \exp(C)$.

**Scaling-to-shift invariance.** Let us show that the only smooth solutions of the functional equation (12) are logarithm-type functions (13).

Indeed, since the function $f(x)$ is differentiable, then the function $y_0(\lambda) = f(\lambda \cdot x) - f(x)$ is differentiable too. Thus, we can differentiate both sides of the equation (12) with respect to $\lambda$. As a result, we get

$$x \cdot f'(\lambda \cdot x) = y_0'(\lambda). \tag{A5}$$

In particular, for $\lambda = 1$, we get

$$x \cdot \frac{df}{dx} = a, \tag{A6}$$

where $a \stackrel{\text{def}}{=} y_0'(1)$. We can separate the variables $x$ and $f$ if we multiply both sides of the equality (A6) by $\dfrac{dx}{x}$, then we get

$$df = a \cdot \frac{dx}{x}. \tag{20}$$

Integrating both sides, we get

$$f(x) = a \cdot \ln(x) + C, \tag{A7}$$

where $C$ is the integration constant.