Need for Shift-Invariant Fractional Differentiation Explains the Appearance of Complex Numbers in Physics

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Abstract

Complex numbers are ubiquitous in physics, they lead to a natural description of different physical processes and to efficient algorithms for solving the corresponding problems. But why this seemingly counterintuitive mathematical construction is so natural here? In this paper, we provide a possible explanation of this phenomenon: namely, we show that complex numbers appear if take into account that some physical system are described by derivatives of fractional order and that a physically meaningful analysis of such derivatives naturally leads to complex numbers.

1 Introduction

Formulation of the problem. In many situations ranging from electromagnetic waves and electric circuits to quantum process, the existing physical description of a process uses complex numbers; see, e.g., [4, 8]. This ubiquity of applications is one of the main reasons why complex numbers – at first glance, a strange and somewhat counterintuitive mathematical construction – are actively studied at schools and at the universities.

But a natural question is: *why* are complex numbers ubiquitous in physics?

What we do in this paper. In this paper, we provide a possible explanation for this ubiquity: namely, we show that complex numbers naturally appear when we consider physical processes that require derivatives of fractional order.

2 Our Explanation

Need for fractional derivatives. Usual physical equations contain first-order, second-order (as in Newton's law), sometimes higher-order derivatives.

But often, there are processes which are naturally described by derivatives of fractional order: e.g., of order 1/2; see, e.g., [1, 2, 3, 5, 6, 7] and references therein.

What are the natural properties of the corresponding fractional differentiation operations D^a of fractional order a?

Linearity: first natural property of fractional differentiation. Similar to the usual differentiation, the fractional derivative of a linear combination should be equal to the similar linear combination of fractional derivatives:

$$D^a(c_1 \cdot f_1 + \ldots + c_n \cdot f_n) = c_1 \cdot D^a(f_1) + \ldots + c_n \cdot D^a(f_n), \tag{1}$$

for all possible numbers $c_1, \ldots, c_n$ and functions $f_1, \ldots, f_n$.

Second natural property of fractional differentiation. For each function $f(t)$, we can define its first derivative – which we will denote by Df, its second derivative – which we will denote by D^2f, etc. For such derivatives, if we apply a-th order derivative to the b-th order one, this is equivalent to applying differentiation $a + b$ times:

$$D^a(D^b f) = D^{a+b} f. \tag{2}$$

It is reasonable to require that this property remains true if we consider fractional values a and b. For example, we should have

$$D^{1/2}(D^{1/2} f) = Df. \tag{2a}$$

Shift: a brief reminder. Another physically reasonable property of fractional derivative is related to the fact that t often means time, and for time, there is no fixed starting point. If instead of the original starting point for measuring time, we select another one which is t_0 moments earlier, then to all original numerical values of time, we add the constant t_0: instead of the original value t, we get a new value $t' = t + t_0$.

In the new units, the description $f(t)$ of the same physical process changes: each moment of time t in the new time scale corresponds to moment $t - t_0$ in the original time scale. Thus, in the new scale, this same physical process is described by a new function $f(t - t_0)$. The corresponding transformation of the function $f(t)$ into a new function $f(t - t_0)$ is known as *shift*:

$$(S_{t_0} f)(t) = f(t - t_0). \tag{3}$$

Shift-invariance: third natural property of fractional differentiation. Since the choice of a starting point for measuring time is just a matter of convention – it does not change any physics, it makes sense to require that fractional derivatives do not change if we apply shift. In other words, if we apply a partial derivative to a shifted function, the result should be the same as when we first differentiate in the original time scale and then shift. In other words, we must have the following equality:

$$D^a(S_{t_0} f) = S_{t_0}(D^a f). \tag{4}$$

One can easily check that the usual differentiation – as well as the operations of taking second, third, etc. derivatives – are, in this sense, shift-invariant.

What can we derive from these properties. Let us consider a function $f_k(t) \stackrel{\text{def}}{=} \exp(k \cdot t)$. The importance of this function is that shifting it is equivalent to multiplying it by a constant:

$$(S_{t_0} f_k)(t) = f_k(t - t_0) = \exp(k \cdot (t - t_0)) =$$

$$\exp(-k \cdot t_0) \cdot \exp(k \cdot t) = \exp(-k \cdot t_0) \cdot f + k(t). \tag{5}$$

Due to shift-invariance, if we denote $g_{a,k}(t) \stackrel{\text{def}}{=} (D^a f_k)(t)$, then the fractional derivative $D^a(S_{t_0} f_k)$ of the shifted function $S_{t_0} f_k$ is equal to the shifted version $S_{t_0} g_{a,k}$ of the function $g_{a,k}(t)$, i.e., to

$$(D^a(S_{t_0} f_k))(t) = (S_{t_0} g_{a,k})(t) = g_{a,k}(t - t_0). \tag{6}$$

On the other hand, since, according to the formula (5), the shifted function $S_{t_0} f_k$ is simply equal to the original function f_k multiplied by a constant $C_{k,t_0} \stackrel{\text{def}}{=} \exp(-k \cdot t_0)$, by linearity, the fractional derivative $D^a(S_{t_0} f_k)$ of the shifted function $S_{t_0} f_k$ is equal to the fractional derivative $g_{a,k} = D^a f_k$ of f_k multiplied by the same constant $C_{k,t_0} = \exp(-k \cdot t_0)$:

$$(D^a(S_{t_0} f_k))(t) = \exp(-k \cdot t_0) \cdot g_{a,k}(t). \tag{7}$$

The formula (6) and (7) describe the same quantity, so their right-hand sides must be equal for all t and for all t_0:

$$g_{a,k}(t - t_0) = \exp(-k \cdot t_0) \cdot g_{a,k}(t). \tag{8}$$

In particular, for every real number s, by taking $t = 0$ and $s = -t_0$, we get

$$g_{a,k}(s) = \exp(k \cdot s) \cdot c(a, k), \tag{9}$$

for some constant $c(a, k) \stackrel{\text{def}}{=} g_{a,k}(0)$. In other words, we conclude that for the function $f_k(t) = \exp(k \cdot t)$, we have

$$(D^a f_k)(t) = c(a, k) \cdot f_k(t). \tag{10}$$

This naturally leads to complex numbers. For $a = 1/2$, the formula (10) leads to

$$D^{1/2} f_k = c(1/2, k) \cdot f_k. \tag{11}$$

For a function $f_k(t) = \exp(k \cdot t)$, its derivative Df_k is equal to $k \cdot \exp(k \cdot t)$, i.e., to $k \cdot f_k$. Due to the above-mentioned second natural property of fractional differentiation, we have

$$D^{1/2}(D^{1/2} f_k) = (Df) = k \cdot f_k. \tag{12}$$

Due to (11), the left-hand side of the formula (12) is equal to

$$D^{1/2}(D^{1/2} f_k) = D^{1/2}(c(1/2, k) \cdot f_k). \tag{13}$$

Due to linearity, we have

$$D^{1/2}(D^{1/2} f_k) = D^{1/2}(c(1/2, k) \cdot f_k) =$$

$$c(1/2, k) \cdot (D^{1/2} f_k) = c(1/2, k) \cdot (c(1/2, k) \cdot f_k) = (c(1/2, k))^2 \cdot f_k. \tag{14}$$

By comparing expressions (12) and (14), we conclude that

$$(c(1/2, k))^2 = k. \tag{15}$$

So, for any decreasing exponential function, with $k < 0$, the only way to define fractional derivative satisfying the above natural properties is to use complex (to be more precise, imaginary) values $c(1/2, k)$, and thus, complex-valued result of fractional differentiation!

Thus indeed, here complex numbers naturally appear. This provides one of the possible explanations for the ubiquity of complex numbers.

Comment. Once we allow complex numbers, everything works. One can show that we then naturally have $c(a, k) = k^a$, i.e.,

$$D^a(\exp(k \cdot t)) = k^a \cdot \exp(k \cdot t). \tag{16}$$

Since usual functions can be represented as linear combinations of exponential functions – this is known as *Laplace transform* – we can thus describe fractional derivative of all regular functions.

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