Why Black-Scholes Equations Are Effective Beyond Their Usual Assumptions: Symmetry-Based Explanation

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Abstract

Nobel-Prize-winning Black-Scholes equations are actively used to estimate the price of options and other financial instruments. In practice, they provide a good estimate for the price, but the problem is that their original derivation is based on many simplifying statistical assumptions which are, in general, not valid for financial time series. The fact that these equations are effective way beyond their usual assumptions leads to a natural conclusion that there must be an alternative derivation for these equations, a derivation that does not use the usual too-strong assumptions. In this paper, we provide such a derivation in which the only substantial assumption is a natural symmetry: namely, scale-invariance of the corresponding processes. Scale-invariance also allows us to describe possible generalizations of Black-Scholes equations, generalizations that we hope will lead to even more accurate estimates for the corresponding prices.

1 Formulation of the Problem

Options as financial instruments: reminder. The whole purpose of financial investment is to grow the available amount of money. For this purpose, investors buy stocks (and other financial instruments) that they believe will rise in price.

However, such activity comes with risks:

- if you invest all the money into the stock that is supposed to grow and it does grow, you indeed gain a big profit;
- but on the other hand, if something happened (and the coronavirus epidemics is a good example of such an unpredictable event) and the stock price goes down, you may lose a lot – or even everything you invested.

A less risky option is not to buy all the stocks right away, but to pay a smaller amount of money for the right (but not the obligation) to buy this stock at a certain price K (called the *strike price*) at a pre-determined future moment of time. If only a few people believe that the price will rise, the current price of the stock is taken as the strike price. If many people believe that the stock will rise – but you are more confident in this rise than many others – the strike price is set higher, according to people's expectation of the future price of this stock. This way:

- If the stock does rise as you expected, and its future price (called *spot price*) S will be higher than K, then you can buy this stock at the strike price, immediately sell it at the current stock price, and get a profit of $S - K$ (minus what you paid for this option) for each stock.
- On the other hand, if the stock price tanks, you do not lose all your money – you simply do not exercise this option, and all you lose if what you paid for this option.

Similar contracts can be bought if an investor has stocks that may go down, but does not want to sell them right away. In this case, an investor can buy an option of selling this stock at some fixed future moment of time at a predetermined ("strike") price.

This may sound like modern financial alchemy, but this idea is actually known since ancient Greece: the famous mathematician, astronomer, and philosopher Thales of Miletus – known as one of the Seven Sages of Antiquity – used this to invest money. In the 17 century, when stock exchanges started appearing in Europe, options became one of the way to minimize the investor's risk. Options have been heavily advertised in Amsterdam in 1688, in London in the 1690s, etc.

Such option contracts are actively used nowadays as well. Since such contracts originated in Europe, they are usually called *European options.* These option remain the main type of options that can be bought over the counter.

In the 20th century, many other types of options have been invented. The most well known are *American options*, in which a buyer can exercise his/her right to buy or sell not only at the predetermined future moment of time, but also at any earlier moment of time. Such options can be bought and sold at special *futures exchanges.*

Terminological comment. Options are a particular case of financial instruments generally known as *derivatives* – because, in contract to stocks and bonds whose value comes directly from the corresponding entity (company or municipality), the value of a derivative comes *indirectly*, from possible trade of stocks or bonds.

Estimating the value of an option. What is a reasonable price of an option? In the 17th century, when options became popular, their prices oscillated widely – just like the stock prices oscillated a lot. There was not much information about different companies, no mathematical tools to process the available information. Such oscillations continued until the 1970s, when mathematical equations for pricing the options first appeared in two papers – a paper [3] by Fischer Black and Myron Scholes and a paper [14] by Robert Merton.

How these formulas were applied: from over-hype to over-criticism to a realistic usage. The corresponding formulas were derived under several simplifying assumptions, assumptions that are rather crude approximations to the actual behavior of the stocks.

However, since no better formulas were known, investors started using these approximate formulas to estimate the value of the options – and those investors who used these formulas got, on average, more profit and less loss than those old-fashioned investors who relied on their intuition. This success led many investors to invest in options, and to trust these formulas too much.

But the formulas remain approximate. If you use an approximate formula to decide which of the stocks to buy, it makes sense to trust the formula: if one stock will rise much higher according to the formula, most probably it will indeed rise higher.

However, if you rely on the formula to be exact you can lose a lot of money – and this is exactly what happened to many investors. Indeed, if a formula predicts that between two seemingly equally valued stocks, one will rise 0.5% more than the other one, and if you trust these computations, you can invest all your money hoping that the first stock will have a higher value – and lose everything if it grows a little slower.

This bad experience led to an over-critical attitude to options (and to derivatives in general), they were blamed for economic crises and bankruptcies and shunned.

Eventually, as often happens with new ideas, the wild attitude oscillations stopped, and people realized that these formulas are a useful tool, however, they also realized that this tool's predictions are approximate. The attitude to the authors of these formulas similarly went from calling them supergeniuses to calling them evil geniuses to calling them good scientists. Once all these oscillations settled down, Myron Scholes and Robert Merton – the two remaining authors of the original equations – received the 1997 Nobel Prize in Economics.

Black-Scholes equations: description, current status and the remaining challenge. Let us first present the Black-Scholes equations in precise terms. These equations describe how the price V of an option changes with time. To be more precise, they describe the dependence $V(t, S)$ of the option's price at moment t on t and on the initial price S of the stock. These equations have the following form:

$$\frac{\partial V}{\partial t} = a \cdot V + b \cdot S \cdot \frac{\partial V}{\partial S} + c \cdot S^2 \cdot \frac{\partial^2 V}{\partial S^2}, \qquad (1)$$

for some coefficient a, b, and c.

In the original Black-Scholes equation, a is equal to the risk-free interest rate, $b = -a$, and $c = -V/2$, where V is the variance of difference between stock prices at moments separated by one time unit. However, this formula can be – and is – used in a more general context, where the parameters a, b, and c have to be determined experimentally.

This formula works reasonably well in many situations. The main theoretical challenge is that this formula works surprisingly too well. Namely, this formula was derived under the simplifying condition that the change in the stock's price can be described by the Brownian motion, with finite-variance (e.g., Gaussian) changes. It is know that the actual changes of stock prices with time are much better described by heavy-tail distributions, in which variance can be infinite. The Gaussian assumption was one of the main reason why previous econometric models could not predict the 2008/09 crisis: under the Gaussian distribution, deviations cannot exceed 6 standard deviations, but such deviations do occur during the crises.

Similarly, the assumptions that the stock follows the Brownian motion – i.e., that there is no correlation between changes in stock prices in two consequent moment of time – is also not true: in reality, a period of rising stock prices is usually followed by yet another period of such rising, and vice versa.

Assumptions are not empirically valid, but the conclusions – the Black-Scholes equations – are empirically valid. This means that there should be other reasons why this equation is valid, there should be another derivation of these equations that does not depend on the current empirically invalid assumptions. This is what we do in this paper: we show how it is possible to derive the Black-Scholes equations without invoking the empirically invalid assumptions.

Another challenge is – what next? As we have mentioned, the Black-Scholes equations are an approximation, how can we come up with a more accurate equation? Our derivation of the equations themselves enables us to propose how such – hopefully more accurate – equations can be derived.

2 Symmetry-Based Derivation of Black-Scholes Equations

General formulation of the problem. We want to describe how the price of the option changes with time, i.e., how the rate of change $\dfrac{\partial V}{\partial t}$ depends of the current price $V(t, S)$ of the option corresponding to the same original stock price S and on the prices $V(t, S')$, $V(t, S'')$, $\ldots$, corresponding to other initial stock price values S', S'', $\ldots$:

$$\frac{\partial V}{\partial t} = f(V(t, S), V(t, S'), V(t, S''), \ldots). \qquad (2)$$

First idea: using smoothness. In most applications, dependencies are smooth, so we can expand each dependence in Taylor series in terms of the unknowns and keep only few first terms in this expansion.

We will use this idea to make two simplifying assumptions.

First simplifying assumption: linearity with respect to V. For each individual stock, the value S is relatively small, and the corresponding price V for the option is also relatively small. Big millions only appear if we trade thousands of stocks.

Since the value V is small, we can safely ignore terms which are quadratic (or of higher order) in terms of V and keep only linear terms in the corresponding expansion. Thus, the general formula (2) becomes a simplified – namely, linearized – formula:

$$\frac{\partial V}{\partial t} = a_0(S) + a(S,S) \cdot V(t,S) + a(S,S') \cdot V(t,S') + a(S,S'') \cdot V(t,S'') + \ldots, \quad (3)$$

for some coefficients $a_0(S)$, $a(S,S')$, $a(S,S'')$, ...

When at the present moments, all the prices are 0s, it is not very reasonable to expect that the options will suddenly become valuable in the future – if they were, their price now would not be 0. Thus, if $V(t,S) = V(t,S') = V(t,S'') = \ldots = 0$, we should have $\dfrac{\partial V}{\partial t} = 0$. If we substitute the values $V(t,S) = V(t,S') = V(t,S'') = \ldots = 0$ into the formula (3), then the requirement that $\dfrac{\partial V}{\partial t} = 0$ takes the form $a_0(S) = 0$. For $a_0(S) = 0$, the formula (3) takes an even simpler form:

$$\frac{\partial V}{\partial t} = a(S,S) \cdot V(t,S) + a(S,S') \cdot V(t,S') + a(S,S'') \cdot V(t,S'') + \ldots \quad (4)$$

Second simplifying assumption: keeping only linear and quadratic terms with respect to S. For the dependence of the option price $V(t,S)$ on the initial stock price S, we can similarly keep only the few first terms in the Taylor expansion. Let us keep only linear and quadratic terms, then we get

$$V(t,S') = V(t,S) + (S'-S) \cdot \frac{\partial V}{\partial S} + \frac{1}{2} \cdot (S'-S)^2 \cdot \frac{\partial^2 V}{\partial S^2},$$

$$V(t,S'') = V(t,S) + (S''-S) \cdot \frac{\partial V}{\partial S} + \frac{1}{2} \cdot (S''-S)^2 \cdot \frac{\partial^2 V}{\partial S^2},$$

etc.

Thus, in terms of dependence on $V(t,S)$, all the terms $V(t,S')$, $V(t,S'')$, ..., in the formula (4) are linear combinations of the value $V(t,S)$ and of its first two partial derivatives with respect to S. Substituting these linear combinations into the formula (4), we arrive at the following formula:

$$\frac{\partial V}{\partial t} = c_0(S) \cdot V + c_1(S) \cdot \frac{\partial V}{\partial S} + c_2(S) \cdot \frac{\partial^2 V}{\partial S^2}, \quad (5)$$

for some coefficients $c_i(S)$ depending on S.

So, to fully determine this formula, it is sufficient to describe how the coefficients $c_i(S)$ depend on S.

Why symmetries? We are uncertain about what dependencies $c_i(S)$ will work best. What general technique can we use to deal with this uncertainty?

There are many techniques for dealing with uncertainty, but, in our opinion, any attempts to predict future events come from the following general idea: looks at similar situations in the past, and predict similar consequences. The important word here is "similar", it means, crudely speaking, that the situation is not exactly the same as it was in the past, but it is similar – in the sense that the situation can obtained from the previous ones by some changes (= transformations) that preserve the main properties of the process.

For example, if we drop an object in several US locations and every time, the object starts falling with the acceleration of 9.81 m/sec^2, then when we drop the same object in Asia, we expect to see the same result. Here, the corresponding transformation is shift in space.

In general, transformations under which the main properties are preserved are known as *symmetries*, and the preservation of the corresponding properties is known as *invariance.* Symmetries and invariances are one of the main tools in modern theoretical physics; see, e.g., [4, 19]. Symmetries can also explain where the traditional physical equations come from (see, e.g., [8, 10]) and why some techniques for processing physical data are empirically successful (see, e.g., [7, 12]).

However, symmetries are very efficient beyond physics too. For example, in fuzzy logic (see, e.g., [1, 6, 13, 17, 18, 20]) and in neural networks, both traditional and deep (see, e.g., [2, 5]), symmetries explain why some versions of the corresponding techniques are empirically successful – and some are not; see, e.g., [9, 11, 15, 16].

What are the symmetries in this case. In physics, many symmetries come from the fact that there is often no fixed size. For example, in the old days – before computer modeling became ubiquitous – aerodynamic properties of the newly designed planes were first tested by testing a scaled-down version of the plane in a wind tunnel. Another example is that after the first atomic bomb was used, physicists around the world immediately determined its power – which the US military classified and did not report in their announcements – by comparing the effects of the corresponding explosion with the effects of well-studied smaller-scale explosions.

In such situations, the choice of a measuring unit is arbitrary, the resulting equations should not change if we simply replace, e.g., meters with centimeters, or, more generally, replace the original measuring unit with a λ times smaller one, for some $\lambda > 0$. After this change, all numerical values of the corresponding quantity x are multiplied by λ: $x \to x' = \lambda \cdot x$. For example, 2 meters becomes $100 \cdot 2 = 200$ centimeters. In general, such a transformation is known as *scaling*, and invariance with respect to this transformation is known as *scale-invariance.*

Similarly, in finances, the selection of a monetary unit is an arbitrary process. We can measure everything in US dollars, we can measure everything in Euros or in Yen – the corresponding equations should not change.

Let us apply scale-invariance. Let us apply the scale-invariance requirement to equation (5). When we change a monetary unit to a λ times smaller one, then both the original stock price S and the option price V get multiplied by λ, i.e.:

- instead of S, we get $S' = \lambda \cdot S$, and
- instead of V, we get $V' = \lambda \cdot V$.

We want to require that the equation (5) be equivalent to the same equation, but described in terms of the re-scaled values S' and V':

$$\frac{\partial V'}{\partial S'} = c_0(S') \cdot V' + c_1(S') \cdot \frac{\partial V'}{\partial S'} + c_2(S) \cdot \frac{\partial^2 V'}{(\partial S')^2}. \tag{6}$$

Substituting $V' = \lambda \cdot V$ into the formula (6) and taking into account that

$$\frac{\partial(\lambda \cdot V)}{\partial S'} = \lambda \cdot \frac{\partial V}{\partial S'}$$

and

$$\frac{\partial^2(\lambda \cdot V)}{(\partial S')^2} = \lambda \cdot \frac{\partial^2 V}{(\partial S')^2},$$

we conclude that

$$\lambda \cdot \frac{\partial V}{\partial t} = c_0(S') \cdot \lambda \cdot V + c_1(S') \cdot \lambda \cdot \frac{\partial V}{\partial S'} + c_2(S') \cdot \lambda \cdot \frac{\partial^2 V}{(\partial S')^2}.$$

Dividing both sides of this formula by λ, we get:

$$\frac{\partial V}{\partial S'} = c_0(S') \cdot V + c_1(S') \cdot \frac{\partial V}{\partial S'} + c_2(S') \cdot \frac{\partial^2 V}{(\partial S')^2}.$$

Substituting $S' = \lambda \cdot S$ into this equation and taking into account that

$$\frac{\partial V}{\partial(\lambda \cdot S)} = \frac{1}{\lambda} \cdot \frac{\partial V}{\partial S}$$

and

$$\frac{\partial^2 V}{(\partial(\lambda \cdot S))^2} = \frac{1}{\lambda^2} \cdot \frac{\partial^2 V}{\partial S^2},$$

we get the following formula:

$$\frac{\partial V}{\partial S'} = c_0(\lambda \cdot S) \cdot V + c_1(\lambda \cdot S) \cdot \frac{1}{\lambda} \cdot \frac{\partial V}{\partial S} + c_2(\lambda \cdot S) \cdot \frac{1}{\lambda^2} \cdot \frac{\partial^2 V}{(\partial S)^2}. \tag{7}$$

The equations (5) and (7) should be equivalent, thus, their right-hand sides mut coincide for all possible values of the function V and of its derivatives. Thus, the coefficients at V and at its derivatives must be equal for all possible values of S and λ.

By comparing the coefficients at V, we conclude that $c_0(\lambda \cdot S) = c_0(S)$ for all λ and S. In particular, for $S = 1$ and $\lambda = x$, we conclude that $c_0(x) = c_0(1)$ for all x, i.e., that the function $c_0(x)$ is a constant. Let us denote this constant by c_0.

By comparing the coefficients at the first derivative of V, we conclude that $c_1(\lambda \cdot S) \cdot \dfrac{1}{\lambda} = c_1(S)$, i.e., equivalently, that $c_1(\lambda \cdot S) = \lambda \cdot c_1(S)$ for all λ and S. In particular, for $S = 1$ and $\lambda = x$, we conclude that $c_1(x) = c_1(1) \cdot x$ for all x, i.e., that $c_1(x) = c_1 \cdot x$, where we denoted $c_1 \stackrel{\text{def}}{=} c_1(1)$.

Similarly, by comparing the coefficients at the second derivative of V, we conclude that $c_2(\lambda \cdot S) \cdot \dfrac{1}{\lambda^2} = c_2(S)$, i.e., equivalently, that $c_2(\lambda \cdot S) = \lambda^2 \cdot c_2(S)$ for all λ and S. In particular, for $S = 1$ and $\lambda = x$, we conclude that $c_2(x) = c_2(1) \cdot x^2$ for all x, i.e., that $c_2(x) = c_2 \cdot x^2$, where we denoted $c_2 \stackrel{\text{def}}{=} c_2(1)$.

Substituting the resulting expressions for the functions $c_i(S)$ into the formula (5), we get the following equation:

$$\frac{\partial V}{\partial t} = c_0 \cdot V + c_1 \cdot S \cdot \frac{\partial V}{\partial S} + c_2 \cdot S^2 \cdot \frac{\partial^2 V}{\partial S^2}. \tag{8}$$

We got the desired derivation. The formula (8) is exactly the Black-Scholes equation. So we indeed provided a derivation of the Black-Scholes equation that does not use any of the previously used too-strong assumptions about the corresponding process – in effect, the only requirement that we used was scale-invariance.

3 Conclusions and Future Work

What we did. To estimate the price of options, practitioners use special Black-Scholes equations, equations whose authors received the 1997 Nobel Price in Economics. The main challenge with these equations is that they work surprisingly well in many practical situations, while their usual derivation is based on many strong statistical assumptions, assumptions which are, in general, not empirically valid. It is therefore desirable to look for an alternative derivation of these equations, derivation that would not depends on these too-strong assumptions.

In this paper, we are addressing this challenge by providing a new derivation of the Black-Scholes equations, a derivation for which the only main assumption is a physically reasonable assumption of scale-invariance. The existence of such derivation makes us more confident that Black-Scholes equations will lead to good estimates in new situations as well.

What next? Scale-invariance not only helps to derive the original (approximate) Black-Scholes equations – it can also help to come up with more general – and hopefully, more accurate – equations. For example, if we allow third order terms in the Taylor expansion in S, then arguments similar to the ones above lead to the following more general equation:

$$\frac{\partial V}{\partial t} = c_0 \cdot V + c_1 \cdot S \cdot \frac{\partial V}{\partial S} + c_2 \cdot S^2 \cdot \frac{\partial^2 V}{\partial S^2} + c_3 \cdot S^3 \cdot \frac{\partial^3 V}{\partial S^3}.$$

Alternatively, if we keep only second order terms in the expansion in S but allow quadratic terms in the expansion in V, then scale-invariance leads to another more general formula:

$$\frac{\partial V}{\partial t} = c_0 \cdot V + c_1 \cdot S \cdot \frac{\partial V}{\partial S} + c_2 \cdot S^2 \cdot \frac{\partial^2 V}{\partial S^2} +$$

$$c_{00} \cdot S^{-1} \cdot V^2 + c_{01} \cdot V \cdot \frac{\partial V}{\partial S} + c_{11} \cdot S^{-1} \cdot \left(\frac{\partial V}{\partial S}\right)^2 +$$

$$c_{02} \cdot V \cdot S \cdot \frac{\partial^2 V}{\partial S^2} + c_{12} \cdot S^2 \cdot \frac{\partial V}{\partial S} \cdot \frac{\partial^2 V}{\partial S^2} + c_{22} \cdot S^3 \cdot \left(\frac{\partial^2 V}{\partial S^2}\right)^2.$$

We hope that some of these equations will lead to more accurate estimates for options.

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