When Can We Be Sure that Measurement Results Are Consistent: 1-D Interval Case and Beyond

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Abstract

In many practical situations, measurements are characterized by interval uncertainty – namely, based on each measurement result, the only information that we have about the actual value of the measured quantity is that this value belongs to some interval. If several such intervals – corresponding to measuring the same quantity – have an empty intersection, this means that at least one of the corresponding measurement results is an outlier, caused by a malfunction of the measuring instrument. From the purely mathematical viewpoint, if the intersection is non-empty, there is no reason to be suspicious, but from the practical viewpoint, if the intersection is too narrow – i.e., almost empty – then we should also be suspicious. To be on the safe side, it is desirable to take the second measurement into account only if we are sufficiently sure that this measurement is not an outlier. In this paper, we describe a natural way to formalize this idea.

1 Formulation of the Problem

Usually, we have duplicate measurement results. Measurement are never 100% accurate and never 100% reliable. A natural way to increase the accuracy and reliability of our information is to perform additional measurements – whether they are measurements of the same quantity or measurements of related quantities, based on which we can estimate the value of the desired quantity.

Which measurement results are outliers: an important problem. The fact that measurement are not 100% reliable means that sometimes measuring instrument malfunction – e.g., get stuck in the previously measured value. If we view such a measurement result as reflecting the true value of the measured quantity, we will get a false impression – and we may make bad decision based on this impression. For example, if the temperature in the chemical reactor starts rising above the optimal level, we need to cool it down to avoid it getting into an ineffective regime or even blowing up. However, if the temperature sensor is stuck in the previously measured (normal) value, we will not notice this potential dangerous development. Similarly, if a distance-measuring sensor in a self-driving vehicle gets stuck in the previous value of the distance from the vehicle to a nearby wall, this malfunction may lead to the vehicle hitting this wall.

In all such cases, it is desirable to decide whether all the measurement results are correct or whether some of them are suspicious – possibly outliers, and if there is a suspicion, which measurement results should we throw away and which retain.

What we do in this paper. In this paper, we analyze this problem in the simplest possible case: when:

- we have only one quantity of interest, i.e., our problem is 1-dimensional, and
- all we know about each of the corresponding measurement errors is the upper bound; in this case, if the measurement result is $\widetilde{x}$, and the upper bound on the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x}-x$ (where x is the actual value) is Δ, then all we know about the actual value x is that it is located in the interval $[\widetilde{x}-\Delta, \widetilde{x}+\Delta]$.

In short, in this paper, we deal with the 1-D interval case, in which several measurements of the same quantity resulted in several intervals $[\underline{x}_i, \overline{x}_i]$ ($i = 1, \ldots, n$) supposedly containing the actual value of the desired quantity.

We also discuss what to do in the general multi-D case.

2 How This Problem Solved Now: Description and Limitations

How this problem is solved now: description. The usual approach is that if the intervals $[\underline{x}_1, \overline{x}_1]$, $\ldots$, $[\underline{x}_n, \overline{x}_n]$ do not have a common point – i.e., if their intersection is empty – then clearly there is a contradiction, some of these intervals are outliers. In this case, we can also decide which intervals are outliers.

In principle, if we have a set of n intervals whose intersection is empty, we could select any subset whose intersection is not empty, and call other intervals outlier. There are usually many such subsets, so it is reasonable to select a subset for which the probability that we have made the right choice is the largest. To estimate the corresponding probabilities, we can have into account that we do not have any a priori reason to be suspicious of any of the intervals. Because of this fact, that each intervals has the same prior probability p_0 of being an outlier. Intervals are independent, so the probability that k intervals are outliers is equal to p_0^k. To maximize this value, we need to minimize the number k of outliers – i.e., equivalently, maximize the number of intervals which are declared correct. Thus, out of all subsets with a non-empty intersection, it is reasonable to select a subset that has the largest possible number of intervals.

Comment. This natural idea does not always lead to a unique solution. For example, if we have three intervals $[-3,-1]$, $[-2,2]$, and $[1,3]$ with an empty intersection, we can select two different two-element subsets:

- we could select $[-3,-1]$ and $[-2,2]$,
- or we could select $[-2,2]$ and $[1,3]$.

This non-uniqueness is not a flaw of a method, it is inherent in the problem. Indeed, the original problem is symmetric with respect to changing the sign, and so, there is no way to select one of the above two solutions without breaking this symmetry.

However, there are serious limitations to this approach.

How this problem is solved now: limitations. Suppose that we make two measurements, both with accuracy $\Delta = 1$. As a result of the first measurement, we got the value 0, which means that the actual value of the measured quantity is somewhere in the interval $[-1,1]$. Let us consider several possible results of the second measurement.

First, a normal case: the second measurement results in a vale $\widetilde{x} = 0.5$. In this case, based on the second measurement, we can conclude that the actual value is in the interval $[0.5-1, 0.5+1] = [0.5, 1.5]$. The actual value should therefore belong to the intersection of the two intervals

$$[-1,1] \cap [0.5, 1.5] = [0.5, 1].$$

Second, let us consider a clearly inconsistent case, when the second measurement resulted in the value $\widetilde{x} = 2.1$. In this case, based on this measurement result, we should conclude that x belongs to the interval $[2.1-1, 2.1+1] = [1.1, 3.1]$, but this new interval does not have any common points with the original interval $[-1,1]$:

$$[-1,1] \cap [1.1, 3.1] = \emptyset.$$

And finally, the case that makes the current approach suspicious: suppose that in the second measurement, we got $\widetilde{x} = 2$. In this case, we conclude that x belongs to the interval $[2-1, 2+1] = [1,3]$, and thus, x belongs to the intersection $[-1,1] \cap [1,3] = \{1\}$. So, we get an exact value $x = 1$. From the purely mathematical viewpoint, there is nothing wrong here, but from the physical viewpoint, this does not make sense: we started with two not very accurate measurements, with accuracy ± 1 comparable with the actual values, and we magically got a very precise result???

What if the measurement result is not 2, but a smaller but close numbers, e.g., $\widetilde{x} = 1.9$. In this case, we get the intersection

$$[-1, 1] \cap [1.9 - 1, 1.9 + 1] = [-1, 1] \cap [0.9, 2.9] = [0.9, 1].$$

Again, mathematically very possible, but from the physical viewpoint, we perform two lousy-quality measurements and suddenly got a ten times better accuracy? Such cases are very suspicious, many physicists and engineers would strongly suggest that one of the measurements was an outlier – probably caused by a mismatch.

Resulting problem. How can we formalize this idea? How can we estimate the probability that the measurement results are inconsistent – and dismiss one of them if this probability exceeds a certain threshold?

What we do in this paper. In this paper, we provide a possible solution to this problem.

3 Let Us Estimate the Probability that Measurements Are Consistent

Setting. Let us first consider the simplest case when we only have two measurements. We know that the first measurement, with measurement result $\widetilde{x}$ and accuracy Δ, is legitimate. We are interested in the probability that the second measurement, with measurement result $\widetilde{x}'$ and accuracy Δ', is also legitimate. Equivalently, we may want to estimate the probability that the second measurement is an outlier – which is 1 minus the first probability. We would then accept the second measurement is legitimate if we are sufficiently sure that it is legitimate – i.e., if the probability of its being legitimate exceeds a given threshold t_0.

In this simplest case, we will be able to find an analytical formula for this probability; in more complex cases, we do not have an explicit formula, but we can use the same methodology and estimate the probability by solving the corresponding probabilistic problem – which can be solved, e.g., by Monte-Carlo techniques.

To make computations easier (and thus clearer), we first consider a realistic case when both measurements have the same accuracy $\Delta' = \Delta$. After that, we extend our analysis to the cases when the second measurement is more accurate ($\Delta' < \Delta$) and when the first measurement is more accurate ($\Delta < \Delta'$).

It is reasonable to use Bayes formula. We have two hypotheses:

- a hypothesis that the second measurement is legitimate; we will denote this hypothesis by L: and

- a hypothesis that the second measurement is an outlier which is inconsistent with the first measurement; let us denote this hypothesis by I.

A priori, before we perform any measurements, we know, from the previous experience, which portion of measurements are outliers. In many practical problems, this prior probability p_0 is about 5-10%; in many other cases, this portion is much lower. So, before we actually perform the second measurement, we have prior probabilities of both hypotheses: $P_0(I) = p_0$ and $P_0(L) = 1 - p_0$.

We want to describe how the probabilities of different hypotheses changes when we learn the measurement result E, i.e., in statistical terms, what are the resulting posterior probabilities $P(I)$ and $P(L)$. In statistics, this change is described by the following *Bayes formula* (see, e.g., [4]:

$$P(L) = \frac{P(E \,|\, L) \cdot P_0(L)}{P(E \,|\, L) \cdot P_0(L) + P(E \,|\, I) \cdot P_0(I)} =$$

$$\frac{P(E \,|\, L) \cdot (1 - p_0)}{P(E \,|\, L) \cdot (1 - p_0) + P(E \,|\, I) \cdot p_0}, \tag{1}$$

where $P(E \,|\, H)$ denotes the probability of observing the result E under hypothesis H.

To use this formula, we need to select an event E and to estimate the probabilities $P(E \,|\, L)$ and $P(E \,|\, I)$.

Estimating $P(E \,|\, I)$. There may be some hope of estimating $P(E \,|\, L)$ because we have some information about possible legitimate measurement results. However, when we say that something is an outlier, we do not introduce any information that can be used for such an estimation. Thus, there is no place to start when estimating the conditional probability $P(E \,|\, I)$; all we know is that this probability – as well as any other probability – is somewhere between 0 and 1.

We want to consider the second measurement legitimate if we are sure that the probability $P(L)$ is greater than or equal the threshold t_0, i.e., if $P(L) \ge t_0$ for all possible values of the conditional probability $P(E \,|\, I)$. For this inequality to be always satisfied, it is sufficient to make sure that this inequality is satisfied for the smallest possible value $P(L)$. From the formula (1), one can see that $P(L)$ decreasing with $P(E \,|\, I)$. Thus, the smallest possible value of $P(L)$ is attained when the conditional probability $P(E \,|\, I)$ attains its largest possible value 1, i.e., when

$$\frac{P(E \,|\, L) \cdot (1 - p_0)}{P(E \,|\, L) \cdot (1 - p_0) + p_0} \ge t_0.$$

Multiplying both sides by the denominator of the left-hand side, we get an equivalent inequality

$$P(E \,|\, L) \cdot (1 - p_0) \ge t_0 \cdot P(E \,|\, L) \cdot (1 - p_0) + t_0 \cdot p_0.$$

Moving all the terms containing $P(E \,|\, L)$ to the left-hand sides, we get

$$P(E \,|\, L) \cdot (1 - p_0) \cdot (1 - t_0) \ge p_0 \cdot t_0,$$

i.e.,
$$P(E\,|\,L) \ge c_0 \stackrel{\text{def}}{=} \frac{p_0 \cdot t_0}{(1-p_0)\cdot(1-t_0)}. \tag{2}$$
For example, if the frequency p_0 of outliers in 5%, and we want to achieve at least 80% confidence t_0, then we should use $c_0 \approx 0.21$.

To use this inequality, we need to be able to estimate $P(E\,|\,L)$. Let us show how to do it.

Estimating $P(E\,|\,L)$: case of two measurements of equal accuracy. As we have mentioned earlier, the situation becomes suspicious when the intersection becomes too narrow, i.e., when the ratio r of the width of the intersection to the width 2Δ of the original interval becomes too low. So, as an appropriate event r, it is reasonable to select an inequality $r \le r_0$ for some r_0.

The ratio r depends on the result of the second measurement which, in its turn, depends on the actual value of the measured quantity. So, to estimate the desired conditional probability, we need to estimate the probabilities of different actual values, and the probabilities of different results of the second measurement.

The actual value can be any number from the interval $[\widetilde{x}-\Delta, \widetilde{x}+\Delta]$. There is no reason to assume that some values from this interval are more probable and some are less probable – it is therefore reasonable to assume that all the values from this interval are equally probable, i.e., that the actual value is uniformly distributed on this interval. This argument – known as *Laplace Indeterminacy Principle* – is widely used in applications of statistics; see, e.g., [3, 4].

Alternatively, we can say that the actual value is equal to $x = \widetilde{x} - \Delta x$, where Δx is uniformly distributed on the interval $[-\Delta, \Delta]$. The result $\widetilde{x}'$ of the second measurement is obtained by adding, to the actual value x, the measurement error $\Delta x'$ of the second measurement. We consider the case when both measurements are equally accurate, with the same upper bound on the measurement error $\Delta' = \Delta$, the measurement error $\Delta x'$ can also take any value from the interval $[-\Delta, \Delta]$. Similarly to Δx, it is therefore reasonable to assume that this measurement error is uniformly distributed on the interval $[-\Delta, \Delta]$.

In terms of the first measurement result $\widetilde{x}$ and measurement errors, the second measurement result $\widetilde{x}'$ has the form $\widetilde{x}' = x + \Delta x' = \widetilde{x} - \Delta x + \Delta x'$, i.e., the form $\widetilde{x}' = \widetilde{x} + d$, where we denoted $d \stackrel{\text{def}}{=} -\Delta x + \Delta x'$. The difference takes a value d if the first measurement error takes some value f and the second measurement error takes the value $s = f + d$. The measurement errors Δx and $\Delta x'$ are independent, so for the probability density $\rho(d)$ of the difference δ we can use the independence-based formula $\rho(d) = \int \rho_1(f) \cdot \rho_2(d+f)\,\mathrm{d}f$, where ρ_i are probability density functions corresponding to different measurement errors. Both measurement errors are uniformly distributed on the interval $[-\Delta, \Delta]$, thus $\rho_1(z) = \rho_2(z) = 1/(2\Delta)$ for all $z \in [-\Delta, \Delta]$. Thus, the integral is over all f for which $-\Delta \le f \le \Delta$ and $-\Delta \le f + d \le \Delta$. In particular, for $d \ge 0$, these two inequalities are equivalent to $-\Delta \le f \le \Delta - d$. The width of this integral is $(\Delta - d) - (-\Delta) = 2\Delta - d$. For each such value f, the integrated expression is equal to $\frac{1}{2\Delta} \cdot \frac{1}{2\Delta} = \frac{1}{4\Delta^2}$. Thus, for $d \geq 0$, the integral is equal to $\rho(d) = \frac{1}{4\Delta^2} \cdot (2\Delta - d)$. A similar formula can be derived for $d \leq 0$, so, in general, we get

$$\rho(d) = \frac{1}{4\Delta^2} \cdot (2\Delta - |d|). \tag{3}$$

Based on the measurement result $\widetilde{x}' = \widetilde{x} + d$, we form an interval $[\widetilde{x}' - \Delta, \widetilde{x}' + \Delta] = [\widetilde{x} + d - \Delta, \widetilde{x} + d + \Delta]$ and take the intersection of this interval with the interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$ coming from the first measurement. For $d \geq 0$, this intersection has the form $[\widetilde{x} + d - \Delta, \widetilde{x} + \Delta]$, and has a width of $2\Delta - d$. A similar formula can be obtained from $d \leq 0$, so, in general, the width of the intersection interval is $2\Delta - |d|$, and the ratio between this width and the original width is $r = \frac{2\Delta - |d|}{2\Delta}$. Thus, the inequality $r \leq r_0$ is equivalent to $2\Delta - |d| \leq r_0 \cdot 2\Delta$, i.e., equivalent to $|d| \geq 2\Delta \cdot (1 - r_0)$. The probability that $r \leq r_0$ can be computed as

$$\int_{-2\Delta}^{-2\Delta \cdot (1 - r_0)} \rho(d)\, \mathrm{d}d + \int_{2\Delta \cdot (1 - r_0)}^{2\Delta} \rho(d)\, \mathrm{d}d.$$

Substituting the above formula for the probability density $\rho(d)$ into this expression and computing the integrals, we conclude that

$$P(E \mid L) = r_0^2. \tag{4}$$

As a result, we arrive at the following conclusion:

So when should we be sure that the second measurement is not an outlier: case of two measurements of the same accuracy. Based on formulas (2) and (4), we make this conclusion is

$$r_0 \geq \sqrt{\frac{p_0 \cdot t_0}{(1 - p_0) \cdot (1 - t_0)}}, \tag{5}$$

where:

- r_0 is the ratio of the width of the intersection interval (obtained after two measurements) to the width of the original interval (corresponding to first measurement only),
- p_0 is the frequency of outliers, and
- t_0 is the desired confidence.

For example, if the frequency p_0 of outliers in 5%, and we want to achieve at least 80% confidence t_0, then we should only consider the cases when $r_0 \geq 0.45$. In other words, if the intersection is more than twice narrower than the original interval, this becomes suspicious.

What if the second measurement is more accurate. So far, we have considered the case when both measurements were equally accurate. Let us now consider the case when the second measurement is more accurate, i.e., when $\Delta' < \Delta$. Let us first describe, for this case, the corresponding probability density function $\rho(d) = \int \rho_1(f) \cdot \rho_2(d+f)\,\mathrm{d}f$. Similarly to the above case, without losing generality, it is sufficient to consider the case when $d \geq 0$.

The product of the two probability density functions ρ_i is different from 0 if and only if both factors are positive, i.e., if and only if $-\Delta \leq f \leq \Delta$ and $-\Delta' \leq f + d \leq \Delta'$. The second inequality is equivalent to $-\Delta' - d \leq f \leq \Delta' - d$. By combining the two double inequalities on f, we conclude that $\max(-\Delta, -\Delta'-d) \leq f \leq \min(\Delta, \Delta'-d)$. Here, $\Delta' < \Delta$, thus $\Delta' - d \leq \Delta' < \Delta$. So, the upper bound for f is always equal to $\Delta' - d$.

In the lower bound, the term $-\Delta' - d$ is larger than Δ when $d \leq \Delta - \Delta'$. In this case, the resulting double inequality on f takes the form $-\Delta' - d \leq f \leq \Delta' - d$. The width of the corresponding interval is $2\Delta'$, thus the overall probability density is equal to $\rho(d) = \dfrac{1}{4\Delta \cdot \Delta'} \cdot (2\Delta') = \dfrac{1}{2\Delta}$.

When $d \geq \Delta - \Delta'$, then $-\Delta \geq -\Delta' - d$, thus the inequalities on f take the form $-\Delta \leq f \leq \Delta' - d$. The width of this interval is $\Delta + \Delta' - d$, thus the probability density is equal to $\rho(d) = \dfrac{1}{4\Delta \cdot \Delta'} \cdot (\Delta + \Delta' - d)$.

When the second measurement result is $\widetilde{x}' = \widetilde{x} + d$, the interval that we build based on this measurement result is $[\widetilde{x}' - \Delta', \widetilde{x}' + \Delta'] = [\widetilde{x} + d - \Delta', \widetilde{x} + d + \Delta']$. We need to form an intersection between this interval and the original interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$. For $d \geq 0$, the lower bound $\widetilde{x} + d - \Delta'$ is always larger than $\widetilde{x} - \Delta$, so the left endpoint of the intersection interval is always $\widetilde{x} + d - \Delta$. The right endpoint is the smallest of the values $\widetilde{x} + d + \Delta'$ and $\widetilde{x} + \Delta$.

- When $d \leq \Delta - \Delta'$, the value $\widetilde{x} + \Delta$ is smaller, so the intersection has the form $[\widetilde{x} + \Delta', \widetilde{x} + d + \Delta']$. This interval has width $2\Delta'$ – which is exactly the width of the second interval, and the ratio of widths is $r = \dfrac{\Delta'}{\Delta}$.

- When $d \geq \Delta - \Delta'$, then the intersection has the form $[\widetilde{x} + d - \Delta', \widetilde{x} + \Delta]$; its width is $\Delta + \Delta' - d$, and thus, the ratio r of its width to the width of the original (first) interval is $r = \dfrac{\Delta + \Delta' - d}{2\Delta}$.

The inequality $r \leq r_0$ is equivalent to $\dfrac{\Delta + \Delta' - d}{2\Delta} \leq r_0$, i.e., equivalently, to $\Delta + \Delta' - d \leq r_0 \cdot 2\Delta$ and $d \geq d_0 \stackrel{\text{def}}{=} \Delta \cdot (1 - 2r_0) + \Delta'$. The probability that $r \leq r_0$ can be thus computed as the integral of the probability density function $\rho(d)$ for $d \geq d_0$ (and d smaller than or equal to its largest possible value $\Delta + \Delta'$). This integral is an area under the corresponding curve, and since the function $\rho(d)$ linearly decreases for larger d, this is just an area of the triangle in which one side is $\Delta + \Delta' - d_0$ (which happens to be equal to $r_0 \cdot 2\Delta$) and the height is the value of $\rho(d_0)$, i.e., the value $\dfrac{\Delta + \Delta' - d_0}{4\Delta \cdot \Delta'} = \dfrac{r_0 \cdot 2\Delta}{4\Delta \cdot \Delta'} = \dfrac{r_0}{2\Delta'}$. The area of the triangle is thus equal to $\dfrac{1}{2} \cdot r_0 \cdot 2\Delta \cdot \dfrac{r_0}{2\Delta'} = \dfrac{1}{2} \cdot r_0^2 \cdot \dfrac{\Delta}{\Delta'}$. We need to double this probability, since, in addition to the values $d \geq 0$, we have exactly same set of values $d \leq 0$. Thus, the overall probability that $r \leq r_0$ is equal to

$$P(E \,|\, L) = r_0^2 \cdot \frac{\Delta}{\Delta'}. \tag{6}$$

When should we be sure that the second measurement is not an outlier: case when the second measurement is more accurate. Based on the formulas (2) and (6), we conclude that we should be sure if

$$r_0 \geq \sqrt{\frac{p_0 \cdot t_0}{(1-p_0)\cdot(1-t_0)}} \cdot \sqrt{\frac{\Delta'}{\Delta}}. \tag{7}$$

Comment. On the qualitative level, this formula makes sense: if the second measurement is more accurate, the resulting accuracy is better than if had two measurements of the same accuracy. So, in general, we get a smaller ratio r – and thus, in this case, smaller r does not necessarily indicate possible inconsistency.

What if the first measurement is more accurate. Let us now consider the remaining case, when the first measurement is more accurate, i.e., when $\Delta < \Delta'$. Let us first describe, for this case, the corresponding probability density function $\rho(d) = \int \rho_1(f) \cdot \rho_2(d+f)\,\mathrm{d}f$. Similarly to the above two cases, without losing generality, it is sufficient to consider the case when $d \geq 0$.

The product of the two probability density functions ρ_i is different from 0 if and only if both factors are positive, i.e., if and only if $-\Delta \leq f \leq \Delta$ and $-\Delta' \leq f + d \leq \Delta'$. The second inequality is equivalent to $-\Delta' - d \leq f \leq \Delta' - d$. By combining the two double inequalities on f, we conclude that $\max(-\Delta, -\Delta' - d) \leq f \leq \min(\Delta, \Delta' - d)$. Here, $\Delta < \Delta'$, thus $-\Delta' - d \leq -\Delta' < -\Delta$. Thus, the lower bound for f is always equal to $-\Delta$.

In the upper bound, the term $\Delta' - d$ is larger than Δ when $d \leq \Delta' - \Delta$. In this case, the resulting double inequality on f takes the form $-\Delta \leq f \leq \Delta$. The width of the corresponding interval is 2Δ, thus the overall probability density is equal to $\rho(d) = \dfrac{1}{4\Delta \cdot \Delta'} \cdot (2\Delta) = \dfrac{1}{2\Delta'}$.

When $d \geq \Delta - \Delta'$, then $\Delta' - d \leq \Delta$, thus the inequalities on f take the form $-\Delta \leq f \leq \Delta' - d$. The width of this interval is $\Delta + \Delta' - d$, thus the probability density is equal to $\rho(d) = \dfrac{1}{4\Delta \cdot \Delta'} \cdot (\Delta + \Delta' - d)$.

When the second measurement result is $\widetilde{x}' = \widetilde{x} + d$, the interval that we build based on this measurement result is $[\widetilde{x}' - \Delta', \widetilde{x}' + \Delta'] = [\widetilde{x} + d - \Delta', \widetilde{x} + d + \Delta']$. We need to form an intersection between this interval and the original interval $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$. For $d \geq 0$, the upper $\widetilde{x} + \Delta$ is always smaller than $\widetilde{x} + d + \Delta'$, so the right endpoint of the intersection interval is always $\widetilde{x} + \Delta$. The left endpoint is the largest of the values $\widetilde{x} + d - \Delta'$ and $\widetilde{x} - \Delta$.

- When $d \le \Delta' - \Delta$, the value $\widetilde{x} - \Delta$ is larger, so the intersection has the form $[\widetilde{x} - \Delta, \widetilde{x} + \Delta]$. This interval has width 2Δ – which is exactly the width of the first interval, and the ratio of widths is 1.
- When $d \ge \Delta - \Delta'$, then the intersection has the form $[\widetilde{x} + d - \Delta', \widetilde{x} + \Delta]$; its width is $\Delta + \Delta' - d$, and thus, the ratio r of its width to the width of the original (first) interval is $r = \dfrac{\Delta + \Delta' - d}{2\Delta}$.

The inequality $r \le r_0$ is equivalent to $\dfrac{\Delta + \Delta' - d}{2\Delta} \le r_0$, i.e., equivalently, to $\Delta + \Delta' - d \le r_0 \cdot 2\Delta$ and $d \ge d_0 \stackrel{\text{def}}{=} \Delta \cdot (1 - 2r_0) + \Delta'$. The probability that $r \le r_0$ can be thus computed as the integral of the probability density function $\rho(d)$ for $d \ge d_0$ (and d smaller than or equal to its largest possible value $\Delta + \Delta'$). This integral is an area under the curve, and since the function $\rho(d)$ linearly decreases for larger d, this is just an area of the triangle in which one side is $\Delta + \Delta' - d_0$ (which happens to be equal to $r_0 \cdot 2\Delta$) and the height is the value of $\rho(d_0)$, i.e., the value $\dfrac{\Delta + \Delta' - d_0}{4\Delta \cdot \Delta'} = \dfrac{r_0 \cdot 2\Delta}{4\Delta \cdot \Delta'} = \dfrac{r_0}{2\Delta'}$. The area of the triangle is thus equal to $\dfrac{1}{2} \cdot r_0 \cdot 2\Delta \cdot \dfrac{r_0}{2\Delta'} = \dfrac{1}{2} \cdot r_0^2 \cdot \dfrac{\Delta}{\Delta'}$. We need to double this probability, since, in addition to the values $d \ge 0$, we have exactly same set of values $d \le 0$. Thus, the overall probability that $r \le r_0$ is equal to

$$P(E \,|\, L) = r_0^2 \cdot \frac{\Delta}{\Delta'}. \tag{8}$$

When should we be sure that the second measurement is not an outlier: case when the first measurement is more accurate. Based on the formulas (2) and (8), we conclude that we should be sure if

$$r_0 \ge \sqrt{\frac{p_0 \cdot t_0}{(1 - p_0) \cdot (1 - t_0)}} \cdot \sqrt{\frac{\Delta'}{\Delta}}. \tag{9}$$

Comment. Here, $\Delta' > \Delta$, so the threshold value r_0 is large. This also makes intuitive sense: if the first measurement was more accurate, then it is highly improbable that would improve the overall accuracy by performing the second, much less accurate measurement. If we perform a lousy quality measurement and mysteriously get a very good result, this is highly suspicious.

What to do in the general case. In the general case of multi-D measurements, we can also have suspicious cases when the after adding one more measurement, the area of possible values of the tuple of the corresponding quantities $q = (q_1, \ldots, q_n)$ decreases too much. Similarly to the 1-D case, we do have any reason to believe that some of possible value of q are more probable than others. It is therefore reasonable to conclude that all the values from this area are equally probable. In this case, the probability to be in a sub-area is proportional to the volume of this sub-area. So, if the volume decreases too much, this is suspicious, and it is a sign that the new measurement may be an outlier; see, e.g., [1, 2].

In the general case, we do not have explicit analytic formulas, but we can run Monte-Carlo simulations with simulated measurement errors uniformly distributed in the corresponding intervals, get the corresponding distribution for the decrease in volume, and when the actual decrease is at the end of this distribution – with probability of $r \le r_0$ too small – we dismiss the new measurement result as a possible outlier.

Acknowledgments

This work was supported by the German Research Foundation (DFG) as a part of the Research Training Group i.c.sens (grant GRK2159) and by the Institute of Geodesy of the Leibniz University of Hannover. It was also supported in part by the US National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science) and HRD-1242122 (Cyber-ShARE Center of Excellence). This paper was written when V. Kreinovich was visiting Leibniz University of Hannover.

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