Crack Growth: Theoretical Explanation of Empirical Formulas

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Abstract

Due to stress, cracks appear in constructions: cracks appear in buildings, bridges, pavements, among other structures. In the long run, cracks need to be repaired. However, our resources are limited, so we need to decide which cracks are more dangerous. For this, we need to be able to predict how different cracks will grow. There are several empirical formulas describing crack growth. In this paper, we show that by using scale invariance, we can provide a theoretical explanation for these empirical formulas.

1 Crack Propagation: Empirical Formulas

Studying crack propagation is important. Under stress, crack appear, and once they appear, they start growing. Cracks in an engine can lead to a catastrophe, cracks in a pavement makes a road more dangerous and prone to accidents.

In the ideal world, each crack should be repaired as soon as it is noticed. This is indeed done in critical situations – e.g., after each flight, the Space Shuttle was thoroughly studied and all cracks were repaired. However, in most other (less critical) situations, for example, in pavement engineering, our resources are limited. It is therefore desirable to predict how the current cracks will grow, so that we will be able to concentrate our limited repair resources on most potentially harmful cracks.

How cracks grow: a general description. In most cases, stress comes in cycles: the engine clearly goes through the cycles, the road segment gets stressed when a vehicle passes through it, etc. Thus, the crack growth is usually expressed by describing how the length a of the pavement changes during a stress cycle at which the stress is equal to some value σ. The increase in length is usually denoted by Δa. So, to describe how a crack grows, we need to find out how Δa depends on a and σ:

$$\Delta a = f(a, \sigma), \tag{1}$$

for some function $f(a, \sigma)$.

Case of very short cracks. The first empirical formula – known as Wöhler law – was proposed to describe how cracks appear. In the beginning, the length a is 0 (or very small), so the dependence on a can be ignored, and we have

$$\Delta a = f(\sigma), \tag{2}$$

for some function $f(\sigma)$. Empirical data shows that this dependence is a power law, i.e., that

$$\Delta a = C_0 \cdot \sigma^{m_0}, \tag{3}$$

for some constants C_0 and m_0.

Practical case of reasonable size cracks: Paris law. Very small cracks are extremely important in critical situations: since there, the goal is to prevent the cracks from growing. In most other practical viewpoint, small cracks are usually allowed to grow, so the question is how cracks of reasonable size grow.

Several empirical formulas have been proposed. In 1963, P. C. Paris and F. Erdogan compared all these formulas with empirical data, and came up with a new empirical formula that best fits the data:

$$\Delta a = C \cdot \sigma^m \cdot a^{m'}. \tag{4}$$

This formula – known as *Paris Law* or *Paris-Erdogan Law* – is still in use; see, e.g., [3, 6].

Usual case of Paris law. Usually, we have $m' = m/2$, in which case the formula (4) takes the form

$$\Delta a = C \cdot \sigma^m \cdot a^{m/2} = C \cdot (\sigma \cdot \sqrt{a}\,)^m. \tag{5}$$

The formula (4) is empirical, but the dependence $m' = m/2$ has theoretical explanations. One of such explanations is that the stress acts randomly at different parts of the crack. According to statistics, the standard deviation s of the sum of n independent variables each of which has standard deviation s_0 is equal to $s = s_0 \cdot \sqrt{n}$; see, e.g., [10]. So, on average, the effect of n independent factors is proportional to $\sqrt{n}$. Thus, for a crack of length a, consisting of a/δ_a independent parts, the overall effect K of the stress σ is proportional to

$$K = \sigma \cdot \sqrt{n} \sim \sigma \cdot \sqrt{a}. \tag{6}$$

This quantity K is known as *stress intensity.* For the power law

$$\Delta a = C \cdot K^m, \tag{4a}$$

this indeed leads to

$$\Delta a = \text{const} \cdot (\sigma \cdot \sqrt{a}\,)^m = \text{const} \cdot \sigma^m \cdot a^{m/2}, \tag{7}$$

i.e., to $m' = m/2$.

Empirical dependence between C and m. In principle, we can have all possible combinations of C and m. Empirically, however, there is a relation between C and m:

$$C = c_0 \cdot b_0^m; \tag{8}$$

see, e.g., [4, 5] and references therein.

Beyond Paris law. As we have mentioned, Paris law is only valid for reasonably large crack lengths a. It cannot be valid for $a = 0$, since for $a = 0$, it implies that $\Delta a = 0$ and thus, that cracks cannot appear by themselves – but they do. To describe the dependence (1) for all possible values a, the paper [2] proposed to use the expression (4) with different values of C, m, and m' for different ranges of a. This worked OK, but not perfectly.

The best empirical fit came from the generalization of Paris law proposed in [8]:

$$\Delta a = C \cdot \sigma^m \cdot \left(a^\alpha + c \cdot \sigma^\beta\right)^\gamma. \tag{9}$$

Empirically, we have $\alpha \approx 1$.

What we do in this paper. In this paper, we provide a theoretical explanation for the empirical formulas (3), (4), and (8), and (9).

Our explanations use the general ideas of scale-invariance, ideas very similar to what is described in [4].

2 Scale Invariance: A Brief Reminder

Scale invariance: main idea. In general, we want to find the dependence $y = f(x)$ of one physical quantity on another one – e.g., for short cracks, the dependence of crack growth on stress. When we analyze the data, we deal with numerical values of these quantities, and numerical values depend on the selection of the measuring unit. For example, if we measure crack length in centimeters, we get numerical values which are 2.54 times larger than if we use inches. In general, if we replace the original measuring unit with a new unit which is λ times smaller, all the numerical values get multiplied by λ: instead of the original value x, we get a new value $x' = \lambda \cdot x$.

In many physical situations, there is no preferred measuring unit. In such situations, it makes sense to require that the dependence $y = f(x)$ remain valid in all possible units. Of course, if we change a unit for x, then we need to appropriately change the unit for y. So the corresponding *scale invariance* requirement takes the following form: for every $\lambda > 0$, there exists a value $\mu(\lambda)$ depending on λ such that, if we have

$$y = f(x), \tag{10}$$

then in the new units

$$y' = \mu(\lambda) \cdot y \tag{11}$$

and

$$x' = \lambda \cdot x, \tag{12}$$

we should have

$$y' = f(x'). \tag{13}$$

Similarly, if we are interested in the dependence $y = f(x_1, \ldots, x_v)$ on several quantities $x_1, \ldots, x_v$, then we should similarly require that for all possible tuples $(\lambda_1, \ldots, \lambda_v)$, there should exist a value $\mu(\lambda_1, \ldots, \lambda_v)$ such that if we have

$$y = f(x_1, \ldots, x_v), \tag{14}$$

then in the new units

$$x'_i = \lambda_i \cdot x_i \qquad (15$$

and

$$y' = \mu(\lambda_1, \ldots, \lambda_v) \cdot y, \tag{16}$$

we should have

$$y' = f(x'_1, \ldots, x'_v). \tag{17}$$

Which dependencies are scale invariant. For a single variable, if we plug in the expressions (11) and (12) into the formula (13), we get

$$\mu(\lambda) \cdot y = f(\lambda \cdot x). \tag{18}$$

If we now plug in the expression for y from formula (10) into this formula, we will conclude that

$$\mu(\lambda) \cdot f(x) = f(\lambda \cdot x). \tag{19}$$

It is known (see, e.g., [1]) that every measurable solution to this functional equation has the form

$$y = C \cdot x^m, \tag{20}$$

i.e., the form of a power law.

Similarly, for functions of several variables, if we plug in the expressions (15) and (16) into the formula (17), we get

$$\mu(\lambda_1, \ldots, \lambda_v) \cdot y = f(\lambda_1 \cdot x_1, \ldots, \lambda_v \cdot x_v). \tag{21}$$

If we now plug in the expression for y from formula (14) into this formula, we will conclude that

$$\mu(\lambda_1, \ldots, \lambda_v) \cdot f(x) = f(\lambda_1 \cdot x_1, \ldots, \lambda_v \cdot x_v). \tag{22}$$

It is known (see, e.g., [**?**]) that every measurable solution to this functional equation has the form

$$y = C \cdot x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}. \tag{23}$$

3 Scale Invariance Explains Wöhler Law and Paris Law

How can we use scale invariance here? It would be nice to apply scale invariance to crack growth. However, we cannot directly use it: indeed, in the above arguments, we assumed that y and x_i are different quantities, measured by different units, but in our case Δa and a are both lengths. What can we do?

To apply scale invariance, we can recall that in all applications, stress is periodic: for an engine, we know how many cycles per minute we have, and for a road, we also know, on average, how many cars pass through the give road segment. In both cases, what we are really interested in is how much the crack will grow during some time interval – e.g., whether the road segment needs repairs right now or it can wait until the next year. Thus, what we are really interested in is not the value Δa, but the value $\dfrac{da}{dt}$ which can be obtained by multiplying Δa by the number of cycles per selected time unit.

Since the quantities $\dfrac{da}{dt}$ and Δa differ by a multiplicative constant, they follow the same laws as Δa – but for $\dfrac{da}{dt}$, we already have different measuring units and thus, we can apply scale invariance.

So, let us apply scale invariance. For the case of one variable, scale invariance leads to the formula (20), which explains Wöhler law.

For the case of several variables we similarly get the formula (23), which explains Paris law (4).

Thus, both Wöhler and Paris laws can indeed be theoretically explained – by scale invariance.

4 Scale Invariance Explains How C Depends on m

Idea. Let us show that scale invariance can also the explain the dependence (8) between the parameters C and m of the Paris law (4a).

Indeed, the fact that the coefficients C and m describing the Paris law are different for different materials means that, to determine how a specific crack will grow, it is not sufficient to know its stress intensity K, there must be some other characteristic z on which Δa depends:

$$\Delta a = f(K, z). \tag{24}$$

Let us apply scale invariance. If we apply scale invariance to the dependence of Δa on K, then we can conclude that this dependence is described by a power law, i.e., that

$$\Delta a(K, z) = C(z) \cdot K^{m(z)}, \tag{25}$$

where, in general, the coefficients $C(z)$ and $m(z)$ may depend on z. It is well known that if we go to log-log scale, i.e., consider the dependence of $\ln(\Delta a)$ on $\ln(K)$, then the dependence becomes linear. Indeed, if we take logarithms of both sides of the equality (25), we conclude that

$$\ln(\Delta a(K, z)) = m(z) \cdot \ln(K) + \ln(C(z)). \tag{26}$$

Similarly, if we apply scale invariance to the dependence of Δa on z, we also get a power law

$$\Delta a(K, z) = C'(K) \cdot z^{m'(K)} \tag{27}$$

for some values $C'(K)$ and $m'(K)$, i.e., in log-log scale,

$$\ln(\Delta a(K, z)) = m'(K) \cdot \ln(z) + \ln(C'(k)). \tag{28}$$

The logarithm $\ln(\Delta a(K, z))$ in linear in $\ln(K)$ and linear in $\ln(z)$, thus it is a bilinear function of $\ln(K)$ and $\ln(z)$. A general bilinear function has the form:

$$\ln(\Delta a(K, z)) = a_0 + a_K \cdot \ln(K) + a_z \cdot \ln(z) + a_{Kz} \cdot \ln(K) \cdot \ln(z), \tag{29}$$

i.e., the form

$$\ln(\Delta a(K, z)) = (a_0 + a_z \cdot \ln(z)) + (a_K + a_{Kz} \cdot \ln(z)) \cdot \ln(K). \tag{30}$$

By applying $\exp(t)$ to both sides of the formula (30), we conclude that the dependence of Δa on K has the form

$$\Delta a = C \cdot K^m, \tag{31}$$

where

$$C = \exp(a_0 + a_z \cdot \ln(z)) \tag{32}$$

and

$$m = a_K + a_{Kz} \cdot \ln(z). \tag{33}$$

From (33), we conclude that $\ln(z)$ is a linear function of m, namely, that

$$\ln(z) = \frac{1}{a_{Kz}} \cdot m - \frac{a_K}{a_{Kz}}. \tag{34}$$

Substituting this expression for $\ln(z)$ into the formula (32), we can conclude that

$$C = \exp\left(\left(a_0 - \frac{a_K \cdot a_z}{a_{Kz}}\right) + \frac{a_z}{a_{Kz}} \cdot m\right), \tag{35}$$

i.e., the desired formula (8), $C = c_0 \cdot b_0^m$, with

$$c_0 = \exp\left(a_0 - \frac{a_K \cdot a_z}{a_{Kz}}\right) \tag{36}$$

and

$$b_0 = \exp\left(\frac{a_z}{a_{Kz}}\right). \tag{37}$$

Thus, the empirical dependence (8) of C on m can also be explained by scale invariance.

5 Scale Invariance Explains Generalized Paris Law

Analysis of the problem. Let us show that scale invariance can also explain the generalized Paris law (9).

So far, we have justified two laws: Wöhler law (3) that describes how cracks appear and start growing, and Paris law (4) that describes how they grow once they reach a certain size. In effect, these two laws describe two different mechanisms for crack growth. To describe the joint effect of these two mechanisms, we need to combine the effects of both mechanisms.

How can we combine the two formulas? If the effect of the first mechanism is denoted by q_1 and the effect of the second one by q_2, then a natural way to combine them is to consider some function

$$q = F(q_1, q_2). \tag{38}$$

What should be the properties of this combination function?

If one the effects is missing, then the overall effect should coincide with the other effect, so we should have $F(0, q_2) = q_2$ and $F(q_1, 0) = q_1$ for all q_1 and q_2.

If we combine two effects, it should not matter in what order we consider them, i.e., we should have

$$F(q_1, q_2) = F(q_2, q_1) \tag{39}$$

for all q_1 and q_2. In mathematical terms, the combination operation $F(q_1, q_2)$ should be *commutative*.

Similarly, if we combine three effects, the result should not depend on the order in which we combine them, i.e., that we should have

$$F(F(q_1, q_1), q_3) = F(q_1, F(q_2, q_3)) \tag{40}$$

for all q_1, q_1, and q_3. In mathematical terms, the combination operation $F(q_1, q_2)$ should be *associative*.

It is also reasonable to require that if we increase one of the effects, then the overall effect will increase, i.e., that the function $F(q_1, q_2)$ should be *strictly monotonic* in each of the variables: if $q_1 < q_1'$, then we should have

$$F(q_1, q_2) < F(q_1', q_2).$$

It is also reasonable to require that small changes to q_i should lead to small changes in the overall effect, i.e., that the function $F(q_1, q_2)$ should be *continuous*.

Finally, it is reasonable to require that the operation $F(q_1, q_2)$ be *scale invariant* in the following sense: if $q = F(q_1, q_2)$, then for every $\lambda > 0$, if we take $q_i' = \lambda \cdot q_i$ and $q' = \lambda \cdot q$, then we should have $q' = F(q_1', q_2')$.

What are the resulting combination functions. It is known – see, e.g., [9] – that every commutative, associative, strictly monotonic, continuous, and scale invariant combination operation for which $F(q_1, 0) = q_1$ has the form

$$F(q_1, q_2) = (q_1^p + q_2^p)^{1/p} \tag{41}$$

for some $p > 0$.

This explains the generalized Paris law. Indeed, if we substitute the expression (3) instead of q_1 and the expression (4) instead of q_2 into the formula (41), we get

$$\Delta a = \left((C_0 \cdot \sigma^{m_0})^p + \left(C \cdot \sigma^m \cdot a^{m'}\right)^p\right)^{1/p} =$$

$$\left(C_0^p \cdot \sigma^{m_0 \cdot p} + C^p \cdot \sigma^{m \cdot p} \cdot a^{m' \cdot p}\right)^{1/p} =$$

$$C \cdot \sigma^m \cdot \left(a^{m' \cdot p} + \left(\frac{C_0}{C}\right)^p \cdot \sigma^{(m - m_0) \cdot p}\right)^{1/p}, \tag{42}$$

i.e., we get the desired formula (9), with $\alpha = m' \cdot p$, $c = \left(\frac{C_0}{C}\right)^p$, $\beta = (m - m_0) \cdot p$, and $\gamma = 1/p$.

Thus, the generalized Paris law can also be explained by scale invariance.

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