Euclidean Distance Between Intervals Is the Only Representation-Invariant One

Olga Kosheleva and Vladik Kreinovich
University of Texas at El Paso
500 W. University
El Paso, TX 79968, USA
olgak@utep.edu, vladik@utep.edu

Abstract

An interval can be represented as a point in a plane, e.g., as a point with its endpoints as coordinates. We can thus define distance between intervals as the Euclidean distance between the corresponding points. Alternatively, we can describe an interval by its center and radius, which leads to a different definition of distance. Interestingly, these two definitions lead, in effect, to the same distance – to be more precise, these two distances differ by a multiplicative constant. In principle, we can have more general distances on the plane. In this paper, we show that only for Euclidean distance, the two representations lead to the same distance between intervals.

1 Formulation of the Problem

Need for interval uncertainty. Most information about physical quantities comes from measurements, and measurements are never absolutely exact: the actual (unknown) value x of a physical quantity is, in general, different from the measurement result $\widetilde{x}$.

In many real-life situations, the only information that we have about the measurement error $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$ is the upper bound Δ on its absolute value: $|\Delta x| \leq \Delta$. In this case, based on the measurement result, the only information that we gain about the actual value x is that this value is somewhere in the interval $[\underline{x}, \overline{x}]$, where $\underline{x} = \widetilde{x} - \Delta$ and $\overline{x} = \widetilde{x} + \Delta$; see, e.g., [1, 2, 3, 4].

Two representations of intervals. In line with the previous subsection, we can have two natural representations of an interval:

- we can represent an interval by its midpoint $\widetilde{x}$ and radius (half-width) Δ;

- alternatively, we can represent an interval by its endpoints $\underline{x} = \widetilde{x} - \Delta$ and $\overline{x} = \widetilde{x} + \Delta$.

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Two representations lead to two natural definitions of the distance between intervals. If we represent an interval by a pair $(\tilde{x}, \Delta)$, then it is natural to define the distance between two intervals as the Euclidean distance between the corresponding 2-D points

$$d((\tilde{x}_1, \Delta_1), (\tilde{x}_2, \Delta_2)) = \sqrt{(\tilde{x}_1 - \tilde{x}_2)^2 + (\Delta_1 - \Delta_2)^2}. \tag{1}$$

On the other hand, if we represent an interval by its endpoints $(\underline{x}, \overline{x})$, then it is natural to define the distance between two intervals as the Euclidean distance between the corresponding 2-D points:

$$D((\underline{x}_1, \overline{x}_1), (\underline{x}_2, \overline{x}_2)) = \sqrt{(\underline{x}_1 - \underline{x}_2)^2 + (\overline{x}_1 - \overline{x}_2)^2}. \tag{2}$$

The two metrics differ by a multiplicative constant. The two metrics differ by a multiplicative constant. Indeed, if we denote $a \stackrel{\text{def}}{=} \tilde{x}_1 - \tilde{x}_2$ and $b \stackrel{\text{def}}{=} \Delta_1 - \Delta_2$, then the first distance has the form

$$d = \sqrt{a^2 + b^2}. \tag{3}$$

In these terms,

$$\underline{x}_1 - \underline{x}_2 = (\tilde{x}_1 - \Delta_1) - (\tilde{x}_2 - \Delta_2) = (\tilde{x}_1 - \tilde{x}_2) - (\Delta_1 - \Delta_2) = a - b, \tag{4}$$

and

$$\overline{x}_1 - \overline{x}_2 = (\tilde{x}_1 + \Delta_1) - (\tilde{x}_2 + \Delta_2) = (\tilde{x}_1 - \tilde{x}_2) + (\Delta_1 - \Delta_2) = a + b. \tag{5}$$

Thus,

$$D = \sqrt{(a - b)^2 + (a + b)^2}. \tag{6}$$

The expression under the square root is equal to

$$(a = b)^2 + (a + b)^2 = 2(a^2 + b^2),$$

thus $D = \sqrt{2} \cdot d$.

Instead of distances, we can consider their squares d^2 and D^2, then we have $D^2 = 2d^2$.

Natural question. In the 2-D plane, instead of the square of the Euclidean distance $(x_1 - x_2)^2 + (y_1 - y_2)^2$, we can consider more general expressions

$$f(x_1 - x_2) + f(y_1 - y_2), \tag{7}$$

for some even function $f(x)$ which is increasing for $x \geq 0$. For example, for $f(x) = |x|^p$, we get ℓ_p-distance.

A natural question is: for which functions $f(x)$, the values of corresponding f-distance (7) corresponding to two representations of an interval always differ by a multiplicative constant?

What we do in this paper. In this paper, we prove that the only functions $f(x)$ with this property are functions of the type $f(x) = c \cdot x^2$ corresponding to the usual Euclidean distance.

2   Main Result

Discussion. In terms of the above-defined differences a and b, the desired property has the following form:

$$f(a + b) + f(a - b) = C \cdot (f(a) + f(b)). \tag{8}$$

Proposition. *The following two conditions are equivalent:*

- *$f(x)$ is an even function which is strictly increasing for $x \geq 0$ and for which there exists a constant C for which (8) holds for all a and b;*

- *$f(x) = c \cdot x^2$ for some constant $c > 0$.*

Proof.

$0°$. Clearly, the function $f(x) = c \cdot x^2$ is an even function which is strictly increasing for $x \geq 0$ and for which there exists a constant $C = 2$ for which (8) holds for all a and b.

So, to prove the proposition, it is sufficient to prove that if a function $f(x)$ is an even function which is strictly increasing for $x \geq 0$ and for which there exists a constant C for which (8) holds for all a and b, then $f(x) = c \cdot x^2$ for some $c > 0$. Let us now assume that $f(x)$ is such a function.

$1°$. Let us first prove that $C = 2$ and $f(0) = 0$.

Indeed, for $b = 0$, the formula (8) takes the form $2f(a) = C \cdot (f(a) + f(0))$, i.e., the form $(2 - C) \cdot f(a) = C \cdot f(0)$. We cannot have $C \neq 2$ since then the expression $(2 - C) \cdot f(a)$ would be either increasing or decreasing and will not be equal to a constant $C \cdot f(0)$. Thus, $C = 2$ and hence, $f(0) = 0$. For $C = 2$, the equality (8) take the form

$$f(a + b) + f(a - b) = 2(f(a) + f(b)). \tag{9}$$

$2°$. Let us prove, by induction, that for every $n \geq 1$, we have

$$f(n \cdot a) = n^2 \cdot f(a). \tag{10}$$

Indeed, for $n = 1$, this is trivially true. So, we have the induction base.

Let us now prove the induction step. Let us assume that we have proved (10) for all $n = 1, \ldots, k$, let us prove that this equality holds for $n = k + 1$ as well. For this, let is take $b = k \cdot a$. Then, the formula (9) takes the form

$$f((k + 1) \cdot a) + f((k - 1) \cdot a) = 2(f(a) + f(k \cdot a)),$$

hence

$$f((k + 1) \cdot a) = 2(f(a) + f(k \cdot a)) - f((k - 1) \cdot a). \tag{11}$$

We already know that
$$f(k \cdot a) = k^2 \cdot f(a), \tag{12}$$
and that
$$f((k-1) \cdot a) = (k-1)^2 \cdot f(a). \tag{13}$$
Substituting (12) and (13) into (11), we get
$$f((k+1) \cdot a) = 2(f(a) + k^2 \cdot f(a)) - (k-1)^2 \cdot f(a) = (2 + 2k^2 - (k-1)^2) \cdot f(a) =$$
$$(2 + 2k^2 - k^2 + 2k - 1) \cdot f(a) = (k^2 + 2k + 1) \cdot f(a) = (k+1)^2 \cdot f(a). \tag{14}$$
So, by induction, the formula (10) is indeed true for all n.

3°. Let us now prove that
$$f(r) = c \cdot r^2 \tag{15}$$
for all rational numbers $r = \dfrac{p}{q}$.

Indeed, for $n = q$ and $a = \dfrac{1}{q}$, the formula (10) implies that

$$f(1) = q^2 \cdot f\left(\frac{1}{q}\right), \tag{16}$$

hence

$$f\left(\frac{1}{q}\right) = c \cdot \left(\frac{1}{q}\right)^2, \tag{17}$$

where we denoted $c \overset{\text{def}}{=} f(1)$.

Now, for $n = p$ and $a = \dfrac{1}{q}$, the formula (10) implies that

$$f\left(\frac{p}{q}\right) = p^2 \cdot f\left(\frac{1}{q}\right). \tag{18}$$

Substituting the expression (17) for $f\left(\dfrac{1}{q}\right)$ into the formula (18), we conclude that

$$f\left(\frac{p}{q}\right) = c \cdot \left(\frac{p}{q}\right)^2. \tag{19}$$

The statement is proven.

4°. To complete the proof, we need to show that the formula $f(x) = c \cdot x^2$ holds for all real values $x \geq 0$. Indeed, for each q, each real number can be approximated, from below and from above, by fractions

$$\frac{p(q)}{q} \leq x \leq \frac{p(q) + 1}{q}, \tag{20}$$

where $p(q) \stackrel{\text{def}}{=} \lfloor q \cdot x \rfloor$. Since the function $f(x)$ is increasing for $x \geq 0$, we have

$$f\left(\frac{p(q)}{q}\right) \leq f(x) \leq f\left(\frac{p(q)+1}{q}\right). \tag{21}$$

Due to Part 3 of this proof, we have

$$c \cdot \left(\frac{p(q)}{q}\right) \leq f(x) \leq c \cdot \left(\frac{p(q)+1}{q}\right)^2. \tag{22}$$

In the limit $q \to \infty$, both the left- and the right-hand sides of this double inequality tend to $c \cdot x^2$, this indeed $f(x) = c \cdot x^2$ for all $x \geq 0$.

Since the function $f(x)$ is even, this equality is true for all r. The proposition is proven.

Acknowledgments

References

[1] L. Jaulin, M. Kiefer, O. Dicrit, and E. Walter, *Applied Interval Analysis*, Springer, London, 2001.

[2] G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.

[3] R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.

[4] S. G. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer, New York, 2005.