How the Amount of Cracks and Potholes Grows with Time: Symmetry-Based Explanation of Empirical Dependencies

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Abstract

Empirical double-exponential formulas are known that describe how the amount of cracks and potholes in a pavement grows with time. In this paper, we show that these formulas can be explained based on natural symmetries (invariances) – such as invariance with respect to changing the measuring unit or invariance with respect to changing a starting point for measuring time.

1 How the Amount of Cracks and Potholes Grows with Time: Empirical Formulas

Cracks and potholes. When a road is built, it is almost perfect – it has only miniature cracks and potholes, not worthy of these names. However, as the road is used, cracks and potholes appear and start growing.

How transportation engineers usually gauge the amount of cracks and potholes. The amount of cracks is usually gauged the overall length C of the longitudal cracks outside the direct wheel path. The amount of potholes is usually gauged by the total area P of potholes.

As the road is used, the quality of the pavement deteriorates, and the values C and P grow. This growth starts at some small values corresponding to the newly built road – age $t = 0$ – and continues growing until they reach the maximum – the undesirable bad state when the whole road is covered by cracks and potholes.

Empirical formulas. According to [3], both growths are described by similar formulas

$$C = a_C \cdot \exp(-b_C \cdot \exp(-c_C \cdot t)); \tag{1}$$

$$P = a_P \cdot \exp(-b_P \cdot \exp(-c_P \cdot t)). \tag{2}$$

What we do in this paper. In this paper, we use natural symmetry ideas to provide a theoretical explanation for these empirical formulas.

2 Symmetry Ideas: A Brief Reminder

Natural transformations. In science and engineering, we are interested in the values of different physical quantities. We describe these quantities in numerical form, but the numerical values of the corresponding quantities depend on the measuring unit – and for some quantities such as temperature or time, also on the starting point.

If we change the measuring unit for length from meters to centimeters, then all numerical values are multiplied by 100: e.g., 2 m becomes $2 \cdot 100 = 200$ cm. In general, if we replace the original measuring unit with a new unit which is λ times smaller, all numerical values are multiplied by λ: $x \to X = \lambda \cdot x$. This numerical transformation is known as *scaling*.

Similarly, if we start measuring time not from our year 0, but – as the French Revolution suggested – with the year 1789 when the revolution started, then from all year values, we should subtract 1789. In general, if we replace the original starting point with the one which is x_0 units before, then we add x_0 to all numerical values: $x \to X = x + x_0$. This numerical transformation is known as *shift*.

Natural symmetries. For most physical quantities, there is no fixed measuring unit – and sometimes no fixed starting point. It is therefore reasonable to require that the dependencies $y = f(x)$ between physical quantities also not depend on the choice of the measuring unit (and possibly on the choice of the starting point). In physics, such invariance is called *symmetry*; see, e.g., [2, 4].

Of course, if we just change the unit and/or starting point for x, to keep the same formula true in the new units, we may need to appropriately change the unit/starting point for y. For example, to preserve the formula $d = v \cdot t$ – that the path is the product of speed and time – when we change the unit for time, we need to appropriately change the unit for speed.

With this is mind, let us describe possible invariant dependencies.

Scaling-to-scaling (sc-sc). Let us first consider the case when the dependence remains the same after we apply scaling both to x and to y. In precise terms, we assume that for every $\lambda > 0$, there exists a value $\mu(\lambda)$ (depending on λ) such that if $y = f(x)$, then $Y = f(X)$, where $X = \lambda \cdot x$ and $Y = \mu(\lambda) \cdot y$. If we plug in the expressions for Y in terms of y and X in terms of x into the formula $Y = f(X)$, we conclude that $f(\lambda \cdot x) = \mu(\lambda) \cdot y$. Here, $y = f(x)$, so we conclude that

$$f(\lambda \cdot x) = \mu(\lambda) \cdot f(x). \quad (3)$$

It is known (see, e.g., [1]) that every measurable dependence $f(x)$ with this property has the form

$$f(x) = A \cdot x^a, \quad (4)$$

for some A and a.

Comment. The general proof is somewhat complicated, but for differentiable dependencies $f(x)$ – and most physical dependencies are differentiable – this is easy to prove. Indeed, if $f(x)$ is differentiable, then the function $\mu(\lambda) = \dfrac{f(\lambda \cdot x)}{f(x)}$ is differentiable too. Thus, we can differentiate both sides of the equation (3) with respect to λ. As a result, we get

$$x \cdot f'(\lambda \cdot x) = \mu'(\lambda) \cdot f(x). \quad (5)$$

In particular, for $\lambda = 1$, we get

$$x \cdot \frac{df}{dx} = a \cdot f, \quad (6)$$

where $a \stackrel{\text{def}}{=} \mu'(1)$. We can separate the variables x and f if we multiply both sides of the equality (6) by $\dfrac{dx}{x \cdot f}$, then we get

$$\frac{df}{f} = a \cdot \frac{dx}{x}. \quad (7)$$

Integrating both sides, we get

$$\ln(f) = a \cdot \ln(x) + C, \quad (8)$$

where C is the integration constant. Applying the function $\exp(z)$ of both sides of the equality (8), we get the desired expression $f(x) = A \cdot x^a$, with $A = \exp(C)$.

Shift-to-scaling (sh-sc). Let us consider the case when the dependence remains the same after we apply shift to x and scaling to y. In this case, for every x_0, there exists a value $\mu(x_0)$ (depending on x_0) such that if $y = f(x)$, then we have $Y = f(X)$, where $X = x + x_0$ and $Y = \mu(x_0) \cdot y$. If we plug in the expressions for Y in terms of y and X in terms of x into the formula $Y = f(X)$, we conclude that $f(x + x_0) = \mu(x_0) \cdot y$. Here, $y = f(x)$, so we conclude that

$$f(x + x_0) = \mu(x_0) \cdot f(x). \quad (9)$$

It is known (see, e.g., [1]) that every measurable dependence $f(x)$ with this property has the form

$$f(x) = A \cdot \exp(a \cdot x), \quad (10)$$

for some A and a.

Comment. If $f(x)$ is differentiable, then the function $\mu(x_0) = \dfrac{f(x + x_0)}{f(x)}$ is differentiable too. Thus, we can differentiate both sides of the equation (9) with respect to x_0. As a result, we get

$$f'(x + x_0) = \mu'(x_0) \cdot f(x). \quad (11)$$

In particular, for $x_0 = 0$, we get

$$\frac{df}{dx} = a \cdot f, \quad (12)$$

where $a \stackrel{\text{def}}{=} \mu'(0)$. We can separate the variables x and f if we multiply both sides of the equality (6) by $\dfrac{dx}{f}$, then we get

$$\frac{df}{f} = a \cdot dx. \quad (13)$$

Integrating both sides, we get

$$\ln(f) = a \cdot x + C, \quad (14)$$

where C is the integration constant. Applying the function $\exp(z)$ of both sides of the equality (14), we get the desired expression $f(x) = A \cdot \exp(a \cdot x)$, with $A = \exp(C)$.

Scaling-to-shift (sc-sh). Let us now consider the case when the dependence remains the same after we apply scaling to x and shift to y. In precise terms, we assume that for every $\lambda > 0$, there exists a value $y_0(\lambda)$ (depending on λ) such that if $y = f(x)$, then $Y = f(X)$, where $X = \lambda \cdot x$ and $Y = y + y_0(\lambda)$. If we plug in the expressions for Y in terms of y and X in terms of x into the formula $Y = f(X)$, we conclude that $f(\lambda \cdot x) = y + y_0(\lambda)$. Here, $y = f(x)$, so we conclude that

$$f(\lambda \cdot x) = f(x) + y_0(\lambda). \quad (15)$$

It is known (see, e.g., [1]) that every measurable dependence $f(x)$ with this property has the form

$$f(x) = a \cdot \ln(x) + C, \quad (16)$$

for some a and C.

Comment. If $f(x)$ is differentiable, then the function $y_0(\lambda) = f(\lambda \cdot x) - f(x)$ is differentiable too. Thus, we can differentiate both sides of the equation (15) with respect to λ. As a result, we get

$$x \cdot f'(\lambda \cdot x) = y_0'(\lambda). \tag{17}$$

In particular, for $\lambda = 1$, we get

$$x \cdot \frac{df}{dx} = a, \tag{18}$$

where $a \stackrel{\text{def}}{=} y_0'(1)$. We can separate the variables x and f if we multiply both sides of the equality (6) by $\dfrac{dx}{x}$, then we get

$$df = a \cdot \frac{dx}{x}. \tag{19}$$

Integrating both sides, we get

$$f(x) = a \cdot \ln(x) + C, \tag{20}$$

where C is the integration constant.

Shift-to-shift (sh-sh). In this case, for every x_0, there exists a value $y_0(x_0)$ such that if $y = f(x)$, then we have $Y = f(X)$, where $X = x + x_0$ and $Y = y + y_0(x_0)$. If we plug in the expressions for Y in terms of y and X in terms of x into the formula $Y = f(X)$, we conclude that $f(x + x_0) = y + y_0(x_0)$. Here, $y = f(x)$, so we conclude that

$$f(x + x_0) = f(x) + y_0(x_0). \tag{21}$$

It is known (see, e.g., [1]) that every measurable dependence $f(x)$ with this property has the form

$$f(x) = a \cdot x + C, \tag{22}$$

for some a and C.

Comment. If $f(x)$ is differentiable, then the function $y_0(x_0) = f(x + x_0) - f(x)$ is differentiable too. Thus, we can differentiate both sides of the equation (9) with respect to x_0. As a result, we get

$$f'(x + x_0) = y_0'(x_0). \tag{23}$$

In particular, for $x_0 = 0$, we get

$$f'(x) = a, \tag{24}$$

where $a \stackrel{\text{def}}{=} y_0'(0)$. Integrating, we get $f(x) = a \cdot x + C$, where C is the integration constant.

3 So How Does Crack or Pothole Amount Depend on Time

What we want: a brief reminder. We want to find the dependence of the quantity q (crack or pothole amount) on time t. We know:

- that the for $t = 0$, the value $q(t)$ is small positive,
- that the value $q(t)$ increases with time, and
- that the value $q(t)$ tends to some large constant value when t increases.

What are possible symmetries here? For crack amount C and for pothole amount P, there is an absolute starting point – 0, when we have no cracks and no potholes. However, it makes sense to use different units of length and different units of area, so scaling makes perfect sense.

For time, as we have mentioned, both shift and scaling make sense.

First idea. If view of the above analysis, let us see if any of the above symmetric dependencies satisfy the desired property.

Since for q, only scaling makes sense, we can only consider two possibilities: sc-sc and sh-sc. Let us consider them one by one.

First idea: sc-sc case. In the sc-sc case, we have $q(t) = A \cdot t^a$. Since we want a non-negative value, we have to take $A > 0$. Since we want $q(t)$ to be increasing with time, we have to take $a > 0$. However, in this case:

- $q(0)$ is zero – while we want it to be positive, and
- $q(t)$ tends to infinity as t increases – while we want it to tend to some constant.

First idea: sh-sc case. In the sh-sc case, we have $q(t) = A \cdot \exp(a \cdot t)$. Again, since we want a non-negative value, we have to take $A > 0$. Since we want $q(t)$ to be increasing with time, we have to take $a > 0$. In this case:

- $q(0)$ is positive, which is exactly what we wanted, but
- $q(t)$ tends to infinity as t increases – while we want it to tend to some constant.

So what do we do? The first idea does not work, so what should we do?

The above arguments about possible dependencies deal with the case when the quantity y directly depend on the time t. However, in our case, cracks and potholes do not directly depend on time: what changes with time is stress, which, in its turn, causes the pavement to crack. In other words, instead of the direct dependence of the quantity q on time:

- we have q depending on some auxiliary quantity z, and

- we have z depending on time t.

For both dependencies $q(z)$ and $z(t)$ we can have symmetry-motivated formulas. Let us see which combinations of these formulas provide the desired properties of the resulting dependence $q(t) = q(z(t))$ – that this value is positive for $t = 0$, increases for $t > 0$, and tends to a finite limit when $t \to \infty$.

Possible options of the $q(z)$ dependence. Since for q, only scaling is possible, for possible dependencies $q(z)$, we have either the sc-sc option $q(z) = A \cdot z^a$ or the sh-sc option $q(z) = A \cdot \exp(a \cdot z)$.

First option $q(z) = A \cdot z^a$. In this option, when $q(z)$ is sc-sc, it does not make sense to consider sh-sc or sc-sc options for $z(t)$, since, as one can check, this will be equivalent to sh-sc or sc-sc symmetry for $q(t)$, and we have already shown that this is not possible. So, to go beyond previously considered options, we need to consider two remaining options for $z(t)$: sh-sh option $z(t) = a_1 \cdot t + C_1$, and sc-sh option $z(t) = a_1 \cdot \ln(t) + C_1$.

In the first case, we have $q(t) = A \cdot z^a = A \cdot (a_1 \cdot t + C_1)^a$. We can equivalently describe it as $q(t) = A_1 \cdot (t + c_2)^a$, where $A_1 = A \cdot (a_1)^a$ and $c_2 = \dfrac{C_1}{a_1}$. The need to have positive values of q implies $A > 0$, the need to have $q(t)$ increasing leads to $a > 0$, but then, for $t \to \infty$, the resulting expression tends to infinity – while we want it bounded.

In the second case, we have $q(t) = A \cdot z^a = A \cdot (a_1 \cdot \ln(t) + C_1)^a$. Similarly to the first case, we can equivalently describe this expression as $q(t) = A_1 \cdot (\ln(t) + c_2)^a$, with $A_1 = A \cdot (a_1)^a$ and $c_2 = \dfrac{C_1}{a_1}$. The need to have positive values of q implies $A > 0$, the need to have $q(t)$ increasing leads to $a > 0$, but then, for $t \to \infty$, the resulting expression also tends to infinity – while we want it bounded.

Second option $q(z) = A \cdot \exp(a \cdot z)$. In this option, when $q(z)$ is sh-sc, it does not make sense to consider sh-sh or sc-sh options for $z(t)$, since, as one can check, this will be equivalent to sh-sc or sc-sc symmetry for $q(t)$, and we have already shown that this is not possible. So, to go beyond previously considered options, we need to consider two remaining options for $z(t)$: sc-sc option $z(t) = A_1 \cdot t^{a_1}$, and sh-sc option $z(t) = A_1 \cdot \exp(a_1 \cdot t)$.

In the first case, $q(t) = A \cdot \exp(a \cdot z) = A \cdot \exp((a \cdot A_1) \cdot t^{a_1})$. The need to have positive values of q implies $A > 0$. The behavior of this expression depends on the sign of the product $a \cdot A_1$.

- If $a \cdot A_1 > 0$, then the need to have $q(t)$ increasing leads to $a_1 > 0$, but then, for $t \to \infty$, the resulting expression tends to infinity – and we want it bounded.

- If $a \cdot A_1 < 0$, then the need to have $q(t)$ increasing leads to $a_1 < 0$, but then, for $t \to 0$, we have $t^{-|a_1|} \to \infty$, hence $(a \cdot A_1) \cdot t^{-|a_1|} \to -\infty$, and $q(t) = A \cdot \exp((a \cdot A_1) \cdot t^{-|a_1|}) \to 0$, but we want the value $q(0)$ to be positive.

So, the only possible case is the second case, when

$$q(t) = A \cdot \exp(a \cdot z) = A \cdot ((a \cdot A_1) \cdot \exp(a_1 \cdot t)),$$

which is exactly the desired formulas (1) and (2).

Conclusion. So, we can conclude that the only symmetry-motivated dependence $q(t)$ for which $q(0) > 0$ and $q(t)$ increases until some finite number is the dependence (1) and (2). Thus, we have indeed justified the empirical dependencies (1) and (2).

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