Kinematic Metric Spaces Under Interval Uncertainty: Towards an Adequate Definition

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What is a kinematic metric: physical introduction. In the physical space, we can define the distance \(d(a, b)\) between two points as the length of the shortest possible path between them. Thus defined distance is symmetric (\(d(a, b) = d(b, a)\)) and satisfies the usual triangle inequality \(d(a, c) \leq d(a, b) + d(b, c)\). The mathematical notion of a metric is a natural generalization of this physical notion.

From the viewpoint of space-time, physical space corresponds to the situation when we take space-time points (“events”) \((a, t_0)\), \((b, t_0)\), etc. corresponding to the same moment of time \(t_0\). In relativity theory, such events cannot causally influence each other.

When an event \(a\) can causally influence an event \(b\) (we will denote this strict order – i.e., irreflexive transitive – relation by \(a < b\)), this influence is implemented by a particle or particles whose trajectories start at \(a\) and end up at \(b\). For each such trajectory, we can measure the proper time of the corresponding particle. In principle, particles can travel as close to the speed of light as possible, in which case the proper time can be as close to 0 as possible – so the smallest proper time over all trajectories is always 0. Interestingly, there is the largest proper time \(\tau(a, b)\) – which corresponds to inertial motion. The corresponding function \(\tau(a, b)\) – defined only when \(a < b\) – satisfies the “anti-triangle” inequality \(\tau(a, c) \geq \tau(a, b) + \tau(b, c)\).

This inequality describes the known twins paradox of relativity theory: when a twin brother who traveled to the stars comes back to Earth, he will be younger than his twin who stayed on Earth: the biological age of the stay-home brother is \(\tau(a, c)\), while the biological age of the astronaut brother is \(\tau(a, c) + \tau(c, b)\), where \(c\) is the moment when the brother reached a faraway star.

A natural generalization of this function is a notion of kinematic metric.
**Kinematic metric: definition.** Let \((X, <)\) be an ordered set. A function \(\tau(a, b)\) – defined for all pairs for which \(a < b\) – is called a *kinematic metric* if all its values are non-negative and it satisfies the anti-triangle inequality.

**Need for interval uncertainty.** All information about the values of a physical quantity \(v\) – including the values of the kinematic metric – comes from measurements. Measurements are never absolutely accurate, so the measurement result \(\bar{v}\) is, in general, different from the actual (unknown) value \(v\): there is a measurement error \(\Delta v = \bar{v} - v\). Often, the only information that we have about the measurement error is an upper bound \(\Delta\) on its absolute value. In this case, the only information that we have about the actual value \(v\) is that this value belongs to the interval \([\underline{v}, \bar{v}]\) defined as \([\bar{v} - \Delta, \bar{v} + \Delta]\).

**Natural question.** Suppose that we have, for all pairs \(a < b\), intervals \([\tau(a, b), \bar{\tau}(a, b)]\), with \(\bar{\tau}(a, b) \geq 0\), obtained from measurement. If all the upper bounds \(\Delta(a, b)\) are correct, then there is a kinematic metric \(\tau(a, b)\) for which \(\tau(a, b) \in [\tau(a, b), \bar{\tau}(a, b)]\) for all \(a < b\). However, if we – as happens – underestimated the measurement errors, we may not have such a function. So, a natural question is: what is the condition on the intervals \([\tau(a, b), \bar{\tau}(a, b)]\) under which such a function \(\tau(a, b)\) exists?

**A seemingly natural idea does not work.** Anti-triangle inequality implies that \(\tau(a, c) \geq \tau(a, b) + \tau(b, c)\) for all \(a < b < c\). So, it may seem that this inequality is the right condition for the existence of the desired kinematic metric \(\tau(a, b)\). However, this inequality does not guarantee the existence of \(\tau(a, b)\): e.g., for \(X = \{a_1 < a_2 < a_3 < a_4\}\) and \([\tau(a_i, a_j), \bar{\tau}(a_i, a_j)] = [1, 2]\) for all \(i < j\), this inequality is satisfied, but the desired function \(\tau(a, b)\) is not possible: indeed, if it existed, we would have \(2 \geq \tau(a_1, a_4) \geq \tau(a_1, a_2) + \tau(a_2, a_3) + \tau(a_3, a_4) \geq 3\), i.e., \(2 \geq 3\).

**Main result.** For an interval-valued function \([\tau(a, b), \bar{\tau}(a, b)]\) defined for all \(a < b\), the existence of the kinematic metric \(\tau(a, b)\) for which always \(\tau(a, b) \in [\tau(a, b), \bar{\tau}(a, b)]\) is equivalent to the condition that \(\bar{\tau}(a_1, a_n) \geq \sum_{i=1}^{n-1} \tau(a_i, a_{i+1})\) for all sequences \(a_1 < \ldots < a_n\).

**Proof: main idea.** If \(\tau(a, b)\) exists, then this inequality is clearly satisfied. Vice versa, if the above condition is satisfied, then we can take \(\tau(a, b) = \sup \left\{ \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}) \right\}\), where the supremum is taken overall all the chains \(a = a_1 < a_2 < \ldots < a_n = b\) that connect \(a\) and \(b\).

**Comment.** We need the above condition for all natural numbers \(n\): if we only require it only for \(n \leq n_0\), this does not guarantee the existence of \(\tau(a, b)\).