Why Homogeneous Membranes Lead to Optimal Water Desalination: A Possible Explanation

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Abstract A recent experiment has shown that out of all possible biological membranes, homogeneous ones proved the most efficient water desalination. In this paper, we show that natural symmetry ideas lead to a theoretical explanation for this empirical fact.

1 Formulation of the Problem

Membranes. One of the most efficient desalinization techniques is the use of biological membranes.

What was believed and what turned out. Traditionally, researchers believed that the efficiency of a membrane is determined by the average values of relevant quantities – such as the average density of the proteins forming the membrane.

It was known that the knowledge of all these average values enables us to only approximately estimate the membrane's efficiency: two membranes with the same average values of the corresponding quantities may have somewhat different efficiencies.

A recent paper used innovative nano-imaging and nano-manipulating techniques to analyze and control the nano-structure of different membranes. The resulting analysis shows that the difference between efficiencies of different membranes with the same average values of the relevant quantities can be explained by the fact that different membranes have different degrees of homogeneity: specifically, homogenized membranes can be 50% more efficient than the usual non-homogeneous ones [1].

How this phenomenon is explained. According to [1], the newly observed phenomenon can be explained by the fact that density fluctuations are detrimental to

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water transport. This paper also mentions that there may be other factors affecting this phenomenon.

What we do in this paper. In this paper, we provide a general symmetry-based explanation for the observed phenomenon.

2 Towards a Precise Formulation of the Problem

What does optimal mean: general reminder. In simple situations, when the quality of an alternative can be described by a single number, optimization is usually straightforward: we select the alternative for which this number is the largest. In many practical cases, however, the situation is more complicated, we have several different numerical characteristics that need to be taken into account.

What is common is all such cases is that we should be able to decide, given two alternatives a and b:

- whether the alternative a is better; we will denote it by $b < a$,
- or the alternative b is better: $a < b$,
- or these two alternatives are of the same quality to the users; we will denote it by

$$a \sim b.$$

We can combine these two relations into a single preference relation $a \leq b$ meaning that either b is better than a or b has the same quality as a. Once we know this combined relation:

- we can reconstruct $a \sim b$ as $(a \leq b)$ & $(b \leq a)$, and
- we can reconstruct $a < b$ as $(a \leq b)$ & $(b \not\leq a)$.

Clearly, $a \leq a$ for all a, i.e., the relation $\leq$ must be reflexive. Also, if $a \leq b$ and $b \leq c$, then we should have $a \leq c$, i.e., the relation $\leq$ should be transitive.

Preference relation should be final. In general, for a given preference relation, we may have several different alternatives which are optimal – in the sense that they are better than (or of the same quality as) any other alternative. For example, we may have several different membranes that are all equally efficient in terms of water desalination. In this case, we can use this non-uniqueness to optimize something else – e.g., select the membrane with the lowest cost or with the longest expected life.

From the mathematical viewpoint, this means that we replace the original preference relation $\leq$ with a new one $\leq'$ in which $a \leq' b$ if and only if:

- either $a < b$,
- or $a \sim b$ and $a \leq_1 b$ for the additional criterion $\leq_1$.

If after that, we still have several optimal alternatives, we can use this non-uniqueness to optimize something else, etc., until we finally get a *final* preference relation – for which there is only one optimal alternative.

Preference relation should be invariant with respect to natural symmetries. In many practical situations, there are natural symmetries, i.e., natural transformations with respect to which the physical situation does not change. For example, if I drop a pen, it will fall down with the acceleration of 9.81 m/sec^2. If I move to another location and repeat the same experiment, I get the same result. In this sense, the situation does not change with shift. Similarly, if I rotate myself by 90 degrees and repeat the experiment, I get the same result, so the situation is invariant with respect to rotations too.

For the membrane, a natural transformation is shift: if we move from one location of the membrane to another one, nothing should change – since all the related processes are local.

If there is a transformation T that does not change the physical situation, then it is reasonable to require that it should not change our preference relation: i.e., if we had $a \le b$ for some alternatives a and b, then for the transformed alternatives Ta and Tb, we should also have $Ta \le Tb$.

Now, we are ready to formulate the problem in precise terms.

3 Formulation of the Problem in Precise Terms and the Resulting Explanation

Definition 1. *Let A be a set. Its elements will be called* alternatives.

- *By a* preference relation *on the set A, we mean a reflexive and transitive binary relation $\le$.*
- *We say that an alternative $a_{\rm opt}$ is* optimal *with respect to a preference relation $\le$ if $a \le a_{\rm opt}$ for all $a \in A$.*
- *We say that a preference relation is* final *if there exists exactly one alternative which is optimal with respect to this relation.*

Definition 2. *Let $T : A \to A$ be an invertible transformation.*

- *We say that an alternative a is T*-invariant *if $T(a) = a$.*
- *We say that the preference relation $\le$ is T*-invariant *if for every two alternatives a and b, $a \le b$ if and only if $T(a) \le T(b)$.*

Proposition 1. *For every final T-invariant preference relation, the optimal alternative $a_{\rm opt}$ is also T-invariant.*

Corollary. In our case, since the physical situation does not change with shift, it is reasonable to assume that the preference relation should also be invariant with respect to shift. Thus, due to Proposition, we conclude that the optimal membrane should also not change if we shift from one point to another. Since every two locations can be transformed into each other by an appropriate shift, this means that the values of all the corresponding quantities – including density – should be the same at all the locations. In other words, this means that the optimal membrane should be homogeneous, which is exactly what the experiments show. Thus, we have indeed showed that natural symmetry requirements explain the latest experimental results.

Proof. The main idea of this proof first appeared in [2].

Let $\leq$ be a final and T-invariant preference relation, and let $a_{\rm opt}$ be the alternative which is optimal with respect to this relation. This means that $a \leq a_{\rm opt}$ for all $a \in A$. In particular, we have $T^{-1}(a) \leq a_{\rm opt}$, where T^{-1} denotes the inverse function to $T(a)$: $b = T^{-1}(a)$ if and only if $a = T(b)$.

Then, due to T-invariance, we conclude that $T(T^{-1}(a)) \leq T(a_{\rm opt})$, i.e., that $a \leq T(a_{\rm opt})$. This is true for all a, so, by definition of an optimal alternative, the alternative $T(a_{\rm opt})$ is optimal. However, the preference relation $\leq$ is final. This means that there exists only one optimal alternative. Therefore, $T(a_{\rm opt}) = a_{\rm opt}$. Thus, the optimal alternative $a_{\rm opt}$ is indeed T-invariant.

The proposition is proven.

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