Each Realistic Continuous Functional Dependence Implies a Relation Between Some Variables: A Theoretical Explanation of a Fuzzy-Related Empirical Phenomenon

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Abstract

In principle, one can have a continuous functional dependence $y = f(x_1, \ldots, x_n)$ for which, for each proper subset of $n + 1$ variable $x_1, \ldots, x_n, y$, there is no relation: i.e., for each selection of n variables out of these $n + 1$, all combinations of these n values are possible. However, for fuzzy operations, there is always some non-trivial relation between y and one of the inputs x_i; for example, for "and"-operations (t-norms) $y = f_\&(x_1, x_2)$, we have $y \le x_1$; for "or"-operations (t-conorms) $y = f_\vee(x_1, x_2)$ we have $x_1 \le y$, etc. In this paper, we prove a general mathematical explanation for this empirical fact.

1 Formulation of the Problem

Empirical fact. In general, it is quite possible to have a continuous functional dependence $y = f(x_1, \ldots, x_n)$ for which, for each proper subset of $n+1$ variable $x_1, \ldots, x_n, y$, there is no relation: i.e., for each selection of n variables out of these $n + 1$, all combinations of these n values are possible.

One can easily check that, e.g., a linear dependence $y = c_1 \cdot x_1 + \ldots + c_n \cdot x_n$ with non-zero coefficients c_i has this property. Indeed, we can select arbitrary values of n variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y$, then we can find the remaining value x_i as

$$x_i = \frac{y - (c_1 \cdot x_1 + \ldots + c_{i-1} \cdot x_{i-1} + c_{i+1} \cdot x_{i+1} + \ldots + c_n \cdot x_n)}{x_i}.$$

However, for fuzzy operations (see, e.g., [1, 2, 3, 4, 5, 7]), there is always some non-trivial relation between y and one of the inputs x_i. For example, for "and"-operations (t-norms) $y = f_\&(x_1, x_2)$, we have $y \le x_1$. For "or"-operations (t-conorms) $y = f_\vee(x_1, x_2)$ we have $x_1 \le y$. For the average $y = \dfrac{x_1 + x_2}{2}$, we have $x_1/2 \le y$, etc.

Natural question. A natural question is whether the above empirical fact is specific for fuzzy operations, or it is a general mathematical fact.

What we do in this paper. In this paper, we prove that it is indeed a general mathematical fact, which is universally valid – if we consider a realistic setting for this question.

2 Formalization of the Problem

In practice, all the values are bounded. In general, numerical values that we process come from measurements – or from expert estimates. Theoretically, we can consider arbitrarily large and arbitrarily small values, but in practice, our abilities to measure are limited. Each measuring instrument has a bounded range of values that it can measure. There are finitely many different types of measuring instruments. So, by using all of them, all we can cover is a union of bounded ranges covered by each of these instruments – which is itself a bounded set. Thus, all possible measured values of each quantity x_i are located on some interval $[\underline{x}_i, \overline{x}_i]$.

We can only distinguish between finitely many values. Measurements are never absolutely accurate; see, e.g., [6]. We can only measure a quantity with some accuracy $\varepsilon > 0$. From this viewpoint, only values which differ by more than ε are distinguishable – in the sense that they correspond to different actual values. Thus, each measurement result is indistinguishable from one of the values $\underline{x}_i$, $\underline{x}_i + \varepsilon$, $\underline{x}_i + 2\varepsilon$, $\ldots$, $\overline{x}_i - \varepsilon$, $\overline{x}_i$.

In other words, for each quantity, there are only finitely many distinguishable values. We can order them as 0-th, 1-st, etc. Let us denote the number of the last element by m. To simplify the description, we can denote these values simply by 0, 1, $\ldots$, m. In these terms, a function $f(x_1, \ldots, x_n)$ takes n values x_i from the set $\{0, 1, \ldots, m\}$ and returns the value $y = f(x_1, \ldots, x_n)$ from the same set $\{0, 1, \ldots, m\}$. So, we arrive at the following definition.

Definition 1. *Let $m \ge 2$ and $n \ge 2$ be integers. By a m-n-*function*, we will mean a function $f : \{0, 1, \ldots, m\}^n \to \{0, 1, \ldots, m\}$.*

What continuity means in this context. Intuitively, continuity means that if one of the inputs changes a little bit, then the value of the function cannot jump, it much also change only a little bit. In our case, a small change means changing the input by 1, and a jump would means that the resulting value of y changes by 2 or more – thus skipping ("jumping over") intermediate values. Thus, we arrive at the following definition.

Definition 2. *We say that an m-n-function $f(x_1, \ldots, x_n)$ is* continuous *if for every i and all possible values $x_1, \ldots, x_n$ and x'_i, if $|x_i - x'_i| \le 1$, then*

$$|f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) - f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n)| \le 1.$$

What does it mean to imply relations. No relation between n variables means that all combinations of these variables are possible under the given functional dependence. Thus, we arrive at the following definitions.

Definition 3. *We say that a tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$ is* consistent *with the functional relation $y = f(x_1, \ldots, x_n)$ if there exists a value x_i for which*

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = y.$$

Definition 4. *We say that an m-n-function $f(x_1, \ldots, x_n)$* does not imply a relation between the variables *if for every i every tuple $(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$ is consistent with the functional relation $y = f(x_1, \ldots, x_n)$.*

Definition 5. *We say that an m-n-function $f(x_1, \ldots, x_n)$* implies a relation between the variables *if there exist an index i and a tuple*

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y)$$

which is not consistent with the functional relation $y = f(x_1, \ldots, x_n)$.

3 Main Result

Proposition 1. *Every continuous m-n-function $f(x_1, \ldots, x_n)$ implies a relation between the variables.*

Proof.

1°. Let us first prove that if for some $x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$ and $x'_i \ne x_i$, we have

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x'_i, x_{i+1}, \ldots, x_n),$$

then the function $f(x_1, \ldots, x_n)$ implied a relation between the variables.

Indeed, since the value $y = f(x_1, \ldots, x_n)$ is uniquely determined by the values of n variables $x_1, \ldots, x_n$, the overall number of the tuples $(x_1, \ldots, x_n, y)$ which are consistent with the given functional dependence $y = f(x_1, \ldots, x_n)$ is equal to the number of all possible n-tuples $(x_1, \ldots, x_n)$; we will call them *x-tuples.* We have $m+1$ possible values of x_1, we have $m+1$ possible values of x_2, etc., so the overall number of n-tuples is equal to $(m+1)^n$.

In principle, there are also $(m+1)^n$ different possible n-tuples

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, y);$$

let us call them *i-tuples*. Each i-tuple which is consistent with the functional relation is uniquely determined by the corresponding x-tuple. Because of the equality

$$f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = f(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n),$$

two different x-tuples

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) \text{ and } (x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$$

lead to the same i-tuple. Thus, the number of different i-tuples which are consistent with the functional relation is smaller than or equal to $(m+1)^n - 1$. On the other hand, there are $(m+1)^n$ possible i-tuples. This means that at least one of the i-tuples is not consistent with the functional relation – and thus, the corresponding function indeed implies the relation between the variables.

2°. Let us now prove that if for some i and for some values $x_1, \ldots, x_{i-1}, \ldots, x_n$, the value $y = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$ is different from 0 and m, then the function $f(x_1, \ldots, x_n)$ implies a relation between the variables.

Indeed, suppose that $0 < y < m$. In general, for m different x_i, we have $m+1$ values $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$. If two of these values coincide, then, due to Part 1 of this proof, f implies a relation between the variables. So, to prove this result, it is sufficient to consider the case when all $m+1$ values $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ are different.

In particular, this means that the value $f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n)$ is different from $y = f(x_1, \ldots, x_{i-1}, 0, x_{i+1}, \ldots, x_n)$. Because of continuity, it has to be equal either to $y-1$ or to $y+1$.

2.1°. Let us first consider the case when $f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = y+1$.

In our case, the next value $f(x_1, \ldots, x_{i-1}, 2, x_{i+1}, \ldots, x_n)$ cannot be equal to y or to $y+1$, and due to continuity, it cannot differ from $y+1$ by more than 1. Thus, we conclude that $f(x_1, \ldots, x_{i-1}, 2, x_{i+1}, \ldots, x_n) = y+2$ and, in general, that $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = x_i + y$. However, this is not possible for $x_i = m$, since in this case, due to $y > 0$, we have $x_i + y = m + y > m$, while all the values of the function f are between 0 and m.

2.2°. If $f(x_1, \ldots, x_{i-1}, 1, x_{i+1}, \ldots, x_n) = y - 1$, then we similarly get

$$f(x_1, \ldots, x_{i-1}, 2, x_{i+1}, \ldots, x_n) = y - 2$$

and, in general, that $f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) = y - x_i$, but this is not possible for $x_i = m$, since in this case, due to $y < m$, we have $y - x_i = y - m < 0$, while all the values of the function f are between 0 and m.

3°. Let us now consider the value $f(0, \ldots, 0)$. Due to Part 2 of this proof, if this value is different from 0 and m, then the function $f(x_1, \ldots, x_n)$ implies a relation between the variables.

Let us now consider the two remaining cases

$$f(0, 0, \ldots, 0) = 0 \text{ and } f(0, \ldots, 0) = m.$$

3.1°. If $f(0,0,\ldots,0)=0$, then, since the function $f(x_1,\ldots,x_n)$ is continuous, the value $f(1,0,\ldots,0)$ must be 1-close to 0, i.e., equal either to 0 or to 1.

- If $f(1,0,\ldots,0)=0$, then $f(0,0,\ldots,0)=f(1,0,\ldots,0)$, and so, due to Part 1 of this proof, the function $f(x_1,\ldots,x_n)$ implies a relation between the variables.
- If $f(1,0,\ldots,0)=1$, then, due to Part 2 of this proof, the function $f(x_1,\ldots,x_n)$ implies a relation between the variables.

3.2°. Similarly, if $f(0,\ldots,0)=m$, then, since the function $f(x_1,\ldots,x_n)$ is continuous, the value $f(1,0,\ldots,0)$ must be 1-close to m, i.e., equal either to m or to $m-1$.

- If $f(1,0,\ldots,0)=m$, then $f(0,0,\ldots,0)=f(1,0,\ldots,0)$, and so, due to Part 1 of this proof, the function $f(x_1,\ldots,x_n)$ implies a relation between the variables.
- If $f(1,0,\ldots,0)=m-1$, then, due to Part 2 of this proof, the function $f(x_1,\ldots,x_n)$ implies a relation between the variables.

4°. In all the cases, the function $f(x_1,\ldots,x_n)$ implies a relation between the variables. The proposition is proven.

4 Auxiliary Result: Case of $m=1$

Analysis of the problem. In the previous section, we considered the case when $m\ge 2$. But what if $m=1$, i.e., the set of all possible values is the binary set $\{0,1\}$? In this case, the answer is somewhat different, because all possible values are 1-close and thus, all 1-n-functions are continuous:

Proposition 2. *Every 1-n-function $f(x_1,\ldots,x_n)$ is continuous.*

Comment. Since for $m=1$, continuity is no longer a restriction, in this cases, there are some functions which do not imply the relation between the variables. These functions are described in the following proposition:

Definition 6.

- *By a* parity function*, we mean a 1-n-function $f(x_1,\ldots,x_n)$ that returns 1 if the number of 1s among n variables $x_1,\ldots,x_n$ is even, and 0 otherwise.*
- *By an* anti-parity function*, we mean a 1-n-function $f(x_1,\ldots,x_n)$ that returns 0 if the number of 1s among n variables $x_1,\ldots,x_n$ is even, and 1 otherwise.*

Proposition 3. *For a 1-n-function $f(x_1,\ldots,x_n)$, the following two conditions are equivalent to each other:*

- *the function $f(x_1, \ldots, x_n)$ does not imply a relation between the variables, and*
- *the function $f(x_1, \ldots, x_n)$ is either a parity function, or an anti-parity function.*

Comment. With respect to logical operations, this means the main result of this paper – that every continuous function implies a relation between the variables – is only true for fuzzy logic (even if we consider only finitely many fuzzy degrees), but it is not true for the traditional 2-valued logic.

Proof.

1°. Let us first prove that both the parity and the anti-parity functions do not imply the relation between the variables.

Indeed, since we know the value $y = f(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$, we know whether the number n_1 of 1s among $x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n$ needs to be even or odd.

1.1°. If the number n_1 has to be even, then:

- if the number of 1s among the known variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ is already even, then we take $x_i = 0$;
- if the number of 1s among the known variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ is odd, then we take $x_i = 1$.

1.2°. If the number n_1 has to be odd, then:

- if the number of 1s among the known variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ is already odd, then we take $x_i = 0$;
- if the number of 1s among the known variables $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n$ is even, then we take $x_i = 1$.

2°. Let us now assume that the function $f(x_1, \ldots, x_n)$ does not imply a relation between the variables. Let us prove that in this case, this function is either the parity function or the anti-parity function. To prove this, let us consider two possible values (0 or 1) of $f(0, \ldots, 0)$.

2.1°. Let us first consider the case when $f(0, \ldots, 0) = 0$. For each i, what is the possible value of $f(0, \ldots, 0, 1, 0, \ldots, 0)$ where we have 1 on the i-th place? If $f(0, \ldots, 0, 1, 0, \ldots, 0) = f(0, \ldots, 0) = 0$, then, due to Part 1 of the proof of Proposition 1, the function $f(x_1, \ldots, x_n)$ implies a relation between the variables, which contradicts to our assumption. Thus, $f(0, \ldots, 0, 1, 0, \ldots, 0) = 1$ for all i.

Similarly, if we add one more 1, we cannot get the same value of the function f, so we get $f(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) = 0$ for all the tuples that have two 1s. Similarly, we can prove that $f(x_1, \ldots, x_n) = 0$ if we have even number of 1s and $f(x_1, \ldots, x_n) = 1$ if we have odd number of 1s, i.e., that $f(x_1, \ldots, x_n)$ is the anti-parity function.

2.2°. Similarly, let us consider the case when $f(0, \ldots, 0) = 1$. For each i, what is the possible value of $f(0, \ldots, 0, 1, 0, \ldots, 0)$ where we have 1 on the i-th place? If $f(0, \ldots, 0, 1, 0, \ldots, 0) = f(0, \ldots, 0) = 1$, then, due to Part 1 of the proof of Proposition 1, the function $f(x_1, \ldots, x_n)$ implies a relation between the variables, which contradicts to our assumption. Thus, $f(0, \ldots, 0, 1, 0, \ldots, 0) = 0$ for all i.

Similarly, if we add one more 1, we cannot get the same value of the function f, so we get $f(0, \ldots, 0, 1, 0, \ldots, 0, 1, 0, \ldots, 0) = 1$ for all the tuples that have two 1s. Similarly, we can prove that $f(x_1, \ldots, x_n) = 1$ if we have even number of 1s and $f(x_1, \ldots, x_n) = 0$ if we have odd number of 1s, i.e., that $f(x_1, \ldots, x_n)$ is the parity function.

3°. The proposition is proven.

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References

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[3] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[4] H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.

[5] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[6] S. G. Rabinovich, *Measurement Errors and Uncertainties: Theory and Practice*, Springer, New York, 2005.

[7] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.