Kinematic Metric Spaces Under Interval Uncertainty: Towards an Adequate Definition*

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Abstract

In the physical space, we define distance between the two points as the length of the shortest path connecting these points. Similarly, in space-time, for every pair of events for which the event a can causally effect the event b, we can define the longest proper time $\tau(a,b)$ over all causal trajectories leading from a to b. The resulting function is known as kinematic metric. In practice, our information about all physical quantities – including time – comes from measurement, and measurements are never absolutely precise: the measurement result $\widetilde{v}$ is, in general, different from the actual (unknown) value v of the corresponding quantity. In many cases, the only information that we have about each measurement error $\Delta v \stackrel{\text{def}}{=} \widetilde{v} - v$ is the upper bound Δ on its absolute value. In such cases, once we get the measurement result $\widetilde{v}$, the only information we gain about the actual value v is that v belongs to the interval $[\widetilde{v} - \Delta, \widetilde{v} + \Delta]$. In particular, we get intervals $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$ containing the actual values of the kinematic metric. Sometimes, we underestimate the measurement errors; in this case, we may not have a kinematic metric contained in the corresponding narrowed intervals – and this will be an indication of such an underestimation. Thus, it is important to analyze when there exists a kinematic metric contained in all the given intervals. In this paper, we provide a necessary and sufficient condition for the existence of such a kinematic metric. For cases when such a kinematic metric exists, we also provide bounds on its values.

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1 What is a kinematic metric: physical introduction

Physical meaning of distance. In the physical space, we can define the distance $d(a,b)$ between two points as the length of the shortest possible path between them. Thus defined distance is symmetric ($d(a,b)=d(b,a)$) and satisfies the usual triangle inequality

$$d(a,c) \le d(a,b) + d(b,c).$$

The mathematical notion of a metric is a natural generalization of this physical notion.

Distance from the viewpoint of space-time. From the viewpoint of space-time, physical space corresponds to the situation when we take space-time points ("events") (a,t_0), (b,t_0), etc. corresponding to the same moment of time t_0.

In relativity theory, such events cannot causally influence each other. Some events a *can* causally influence events b. For example, in special relativity, an event a can influence the event b if and only if one can go from a to b with a speed not exceeding the speed of light; see, e.g., [2, 5].

In general, the relation "a can causally influence b" – which we will denote by $a<b$ – is irreflexive ($a \not< a$) and transitive. Such relations are known as *strict orders*.

Towards a space-time analog of distance. The causal influence is implemented by a particle (or particles) whose trajectories start at a and end up at b. For each such trajectory, we can measure the proper time of the corresponding particle(s).

A seemingly natural idea is – similarly to how we defined distance – to define its space-time analog as the smallest proper time of all trajectories leading from a to b. However, this idea does not work. Indeed, in principle, particles can travel as close to the speed of light as possible. In this case, the proper time can be as close to 0 as possible. So the *smallest* proper time over all trajectories is always 0.

Good news is that there is the *largest* proper time $\tau(a,b)$ – which corresponds to inertial motion; see, e.g., [2, 5]. The corresponding function $\tau(a,b)$ – defined only when $a<b$ – satisfies the following "anti-triangle" inequality:

$$\tau(a,c) \ge \tau(a,b) + \tau(b,c). \qquad (1)$$

Indeed, $\tau(a,c)$ is the largest proper time over *all* trajectories leading from a to c. In particular, this class includes all trajectories passing through b – and among such trajectories, the longest proper time is $\tau(a,b)+\tau(b,c)$.

The physical meaning of anti-triangle inequality. The inequality (1) describes the known *twins paradox* of relativity theory: if one of the twin brothers travels into space while his brother stays on Earth, then, when a twin brother who traveled to the stars comes back to Earth, he will be younger than his twin who stayed on Earth. Indeed:

- the biological age of the stay-home brother is $\tau(a,c)$, while
- the biological age of the astronaut brother is $\tau(a,b)+\tau(b,c)$, where b is the moment when the brother reached a faraway star.

The notion of a kinematic metric. A natural generalization of the function describing the largest proper time is a notion of *kinematic metric* (see, e.g., [3]):

Definition 1. *Let* $(X, <)$ *be an ordered set. A function* $\tau(a, b)$ *– defined for all pairs for which* $a < b$ *– is called a* kinematic metric *if:*

- *all its values are non–negative and*
- *it satisfies the "anti-triangle" inequality* (1).

2 Need to take interval uncertainty into account

All information about the values of a physical quantity v – including the values of the kinematic metric – comes from measurements. Measurements are never absolutely accurate; see, e.g.. [4]. So the measurement result $\widetilde{v}$ is, in general, different from the actual (unknown) value v: there is a measurement error $\Delta v \stackrel{\text{def}}{=} \widetilde{v} - v$.

Often, the only information that we have about the measurement error is an upper bound Δ on its absolute value; see, e.g., [4]. In this case, the only information that we have about the actual value v is that this value belongs to the interval

$$[\underline{v}, \overline{v}] \stackrel{\text{def}}{=} [\widetilde{v} - \Delta, \widetilde{v} + \Delta].$$

3 A natural question

Suppose that we have, for all pairs $a < b$, intervals

$$[\underline{\tau}(a, b), \overline{\tau}(a, b)] = [\widetilde{\tau}(a, b) - \Delta(a, b), \widetilde{\tau}(a, b) + \Delta(a, b)]$$

with $\underline{\tau}(a, b) \geq 0$, obtained from measurements. If all the upper bounds $\Delta(a, b)$ are correct, then there exists a kinematic metric $\tau(a, b)$ for which $\tau(a, b) \in [\underline{\tau}(a, b), \overline{\tau}(a, b)]$ for all $a < b$.

However, if we – as happens – underestimated the measurement errors, we may not have such a kinematic metric contained in the corresponding narrowed intervals – and this will be an indication of such an underestimation. Thus, it is important to analyze when there exists a kinematic metric contained in all the given intervals. So, a natural question is: what is the condition on the intervals $[\underline{\tau}(a, b), \overline{\tau}(a, b)]$ under which such these intervals contain a kinematic metric $\tau(a, b)$?

4 A seemingly natural idea and why it does not work

Anti-triangle inequality (1) implies that

$$\overline{\tau}(a, c) \geq \underline{\tau}(a, b) + \underline{\tau}(b, c)$$

for all $a < b < c$. So, it may seem that this inequality is the right condition for the existence of the desired kinematic metric $\tau(a, b)$.

However, this inequality does not guarantee the existence of the desired kinematic metric $\tau(a, b)$. For example, for $X = \{a_1 < a_2 < a_3 < a_4\}$ and $[\underline{\tau}(a_i, a_j), \overline{\tau}(a_i, a_j)] = [1, 2]$ for all $i < j$:

- this inequality is satisfied, but
- the desired kinematic metric $\tau(a, b)$ cannot exist.

Indeed, if such a kinematic metric $\tau(a,b)$ existed, we would have:

$$2 \geq \tau(a_1,a_4) \geq \tau(a_1,a_2)+\tau(a_2,a_3)+\tau(a_3,a_4) \geq 3,$$

i.e., $2 \geq 3$.

5 Main result

Proposition 1. *For an interval-valued function $[\underline{\tau}(a,b),\overline{\tau}(a,b)]$ defined for all $a<b$, the following two conditions are equivalent to teach other:*

- *there exists a kinematic metric $\tau(a,b)$ for which $\tau(a,b)\in[\underline{\tau}(a,b),\overline{\tau}(a,b)]$ for all $a<b$; and*
- *for all sequences $a_1<\ldots<a_n$, we have:*

$$\overline{\tau}(a_1,a_n) \geq \sum_{i=1}^{n-1} \underline{\tau}(a_i,a_{i+1}). \tag{2}$$

Comment. This result is analogous to a similar result about interval-valued metrics published in [1].

Proof. If there exists a kinematic metric $\tau(a,b)\in[\underline{\tau}(a,b),\overline{\tau}(a,b)]$, then the inequality (2) is satisfied. Indeed, the inequality (2) follows from the anti-triangle inequality:

$$\overline{\tau}(a_1,a_n) \geq \tau(a_1,a_n) \geq \sum_{i=1}^{n-1} \tau(a_i,a_{i+1}) \geq \sum_{i=1}^{n-1} \underline{\tau}(a_i,a_{i+1}).$$

Vice versa, suppose that the condition (2) is satisfied for all the increasing sequences a_i. Then, we can take

$$\tau(a,b) = \sup\left\{\sum_{i=1}^{n-1} \underline{\tau}(a_i,a_{i+1})\right\}, \tag{3}$$

where the supremum is taken over all the chains $a=a_1<a_2<\ldots<a_n=b$ that connect a and b.

One can easily prove that thus defined function satisfies the anti-triangle inequality. Let us prove that the function (3) satisfies the condition $\tau(a,b)\in[\underline{\tau}(a,b),\overline{\tau}(a,b)]$. Indeed:

- By taking a chain $a_1=a<a_2=b$, we get $\tau(a,b)\geq\underline{\tau}(a,b)$.
- From the inequality (2), for each chain, we get

$$\overline{\tau}(a,b) \geq \sum_{i=1}^{n-1} \underline{\tau}(a_i,a_{i+1}).$$

  Since $\overline{\tau}(a,b)$ is greater than or equal to each sum $\sum\limits_{i=1}^{n-1} \underline{\tau}(a_i,a_{i+1})$, it is greater than or equal to their supremum:

$$\overline{\tau}(a,b) \geq \sup\left\{\sum_{i=1}^{n-1} \underline{\tau}(a_i,a_{i+1})\right\},$$

  i.e, the function (3) satisfies the inequality $\tau(a,b)\leq\overline{\tau}(a,b)$.

Thus, indeed $\underline{\tau}(a,b) \le \tau(a,b) \le \overline{\tau}(a,b)$, i.e., $\tau(a,b) \in [\underline{\tau}(a,b), \overline{\tau}(a,b)]$. The Proposition is proven.

Comment. We need the condition (2) for *all* natural numbers n. Indeed, if we only require (2) for $n \le n_0$, this does not guarantee the existence of $\tau(a,b)$. Here is a counter-example:

- $X = \{a_1 < \ldots < a_{n_0+1}\}$ and
- $[\underline{\tau}(a_i,a_j), \overline{\tau}(a_i,a_j)] = [1, n_0 - 1]$ for all $i < j$.

In this case, there is no kinematic metric $\tau(a,b)$ for which $\tau(a,b) \in [\underline{\tau}(a,b), \overline{\tau}(a,b)]$.. Indeed, if such a kinematic metric $\tau(a,b)$ existed, we would have

$$n_0 - 1 \ge \tau(a_1, a_{n_0+1}) \ge \tau(a_1,a_2) + \ldots + \tau(a_{n_0}, a_{n_0+1}) \ge n_0,$$

i.e., $n_0 - 1 \ge n_0$.

6 First auxiliary result

If for a given interval-valued function $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$, there exists a kinematic metric whose values are contained in these intervals, then a natural next question is: for each pair $a < b$, what are the possible values of $\tau(a,b)$ for such a kinematic metric? Of course, all these values are within the corresponding interval $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$, but, due to the anti-triangle inequality, we can have tighter bounds on $\tau(a,b)$L

Proposition 2. *Let $[\underline{\tau}(a,b), \overline{\tau}(a,b)]$ be an interval-valued function that contains a kinematic metric, and let us consider the class C of all kinematic metrics τ for which $\tau(a,b) \in [\underline{\tau}(a,b), \overline{\tau}(a,b)]$ for all a and b. Then, for each pair $x < y$, we have*

$$\min_{\tau \in C} \tau(x,y) = \tau_m(x,y) \stackrel{\text{def}}{=} \sup \left\{ \sum_{i=1}^{n-1} \underline{\tau}(a_i, a_{i+1}) : x = a_1 < a_2 < \ldots < a_n = y \right\}$$

and

$$\max_{\tau \in C} \tau(x,y) = \tau_M(x,y) \stackrel{\text{def}}{=} \min_{p,q:\ p \le x < y \le q} (\overline{\tau}(p,q) - \tau_m(p,x) - \tau_m(y,q)).$$

Proof. Since the given interval-valued function contains a kinematic metric, the condition (2) is satisfied for this interval-valued function. According to Proposition 1, the value $\tau(x,y) = v$ is possible if and only if the conditions (2) continue to be satisfied when we replace, in the given interval-valued function, the interval $[\underline{\tau}(x,y), \overline{\tau}(x,y)]$ (that corresponds to the pair $x < y$) with the "degenerate" interval $[v,v]$ describing a single value v.

With this replacement, the inequalities of type (2) that do not contain the values $\underline{\tau}(x,y)$ and $\overline{\tau}(x,y)$ remain intact and thus, remain valid. The only inequalities of type (2) that we need to check are inequalities that contain $\underline{\tau}(x,y)$ or $\overline{\tau}(x,y)$: these bounds are now replaced by the value v.

Inequalities that contain $\overline{\tau}(x,y)$ have the form

$$v = \overline{\tau}(x,y) \ge \sum_{i=1}^{n-1} \tau(a_i, a_{i+1}),$$

where $a_1 = x < a_2 < \ldots < a_n = y$. Thus, we conclude that v should be larger than or equal to the largest of these sums, i.e., that v is larger than or equal to $\tau_m(x,y)$.

Inequalities of type (2) that contain $\underline{\tau}(x,y)$ have the form

$$\overline{\tau}(p,q) \ge \sum_{i=1}^{k-1} \underline{\tau}(a_i, a_{i+1}) + \underline{\tau}(x,y) + \sum_{j=1}^{k-1} \underline{\tau}(b_j, b_{j+1}),$$

where $a_1 = p < a_2 < \ldots < a_n = x < y = b_1 < b_2 < \ldots < b_m = q$. Thus,

$$v = \underline{\tau}(x,y) \le \overline{\tau}(p,q) - \sum_{i=1}^{k-1} \underline{\tau}(a_i, a_{i+1}) - \sum_{j=1}^{k-1} \underline{\tau}(b_j, b_{j+1}).$$

For each $p < q$, these inequalities are equivalent to v being smaller than or equal to the smallest of the right-hand sides, i.e., the difference between $\overline{\tau}(p,q)$ and the largest possible values of the sums $\sum\limits_{i=1}^{k-1} \underline{\tau}(a_i, a_{i+1})$ and $\sum\limits_{j=1}^{k-1} \underline{\tau}(b_j, b_{j+1})$. By definition of the function $\tau_m(a,b)$, these largest values are equal to $\tau_m(p,x)$ and $\tau_m(y,q)$. Thus, for each p and q, all the corresponding inequalities are equivalent to

$$v \le \overline{\tau}(p,q) - \tau_m(p,x) - \tau_m(y,q).$$

All these inequalities are equivalent to v being smaller than or equal to the smallest of these differences, i.e., to $v \le \tau_M(x,y)$.

The proposition is proven.

7 Second auxiliary result

A similar result can be proven for the usual metric. Namely, as shown in [1], for a symmetric interval-valued function $[\underline{\rho}(a,b), \overline{\rho}(a,b)]$, the existence of a metric $\rho(a,b)$ for which always $\rho(a,b) \in [\underline{\rho}(a,b), \overline{\rho}(a,b)]$ is equivalent

$$\underline{\rho}(a_1, a_n) \le \sum_{i=1}^{n-1} \overline{\rho}(a_i, a_{i+1}) \qquad (4)$$

for all sequences $a_1, \ldots, a_n$. We can ask a similar question: for each two points a and b, what are the possible values of $\rho(a,b)$ for such a metric? Of course, all these values are within the corresponding interval $[\underline{\rho}(a,b), \overline{\rho}(a,b)]$, but, due to the triangle inequality, we can have tighter bounds on $\rho(a,b)$.

Proposition 3. *Let $[\underline{\rho}(a,b), \overline{\rho}(a,b)]$ be a symmetric interval-valued function that contains a metric, and let us consider the class C of all metrics ρ for which $\rho(a,b) \in [\underline{\rho}(a,b), \overline{\rho}(a,b)]$ for all a and b. Then, for each two points x and y, we have*

$$\max_{\rho \in C} \rho(x,y) = \rho_M(x,y) \stackrel{\text{def}}{=} \inf \left\{ \sum_{i=1}^{n-1} \overline{\rho}(a_i, a_{i+1}) : x = a_1, a_2, \ldots, a_n = y \right\}$$

and

$$\min_{\rho \in C} \rho(x,y) = \max_{p,q} (\underline{\rho}(p,q) - \rho_M(p,x) - \rho_M(y,q)).$$

Proof. Since the given interval-valued function contains a metric, the condition (4) is satisfied for this interval-valued function. According to the above-cited result from

[1], the value $\rho(x,y)=v$ is possible if and only the conditions (4) continue to be satisfied if we replace, in the given interval-valued function, the interval $[\underline{\rho}(x,y),\overline{\rho}(x,y)]$ corresponding to the pair $(x,y)y$, with the "degenerate" interval $[v,v]$ describing a single value v.

With this replacement, the inequalities of type (4) that do not contain the values $\underline{\rho}(x,y)$ and $\overline{\rho}(x,y)$ remain intact and thus, remain valid. The only inequalities of type (4) that we need to check are inequalities that contain $\underline{\rho}(x,y)$ or $\overline{\rho}(x,y)$: these bounds are now replaced by the value v.

Inequalities that contain $\underline{\rho}(x,y)$ have the form

$$v=\underline{\rho}(x,y)\le\sum_{i=1}^{n-1}\rho(a_i,a_{i+1}),$$

where $a_1=x$, a_2, ..., $a_n=y$. Thus, we conclude that v should be smaller than or equal to the smallest of these sums, i.e., that v is smaller than or equal to $\rho_M(x,y)$.

Inequalities of type (4) that contain $\overline{\rho}(x,y)$ have the form

$$\underline{\rho}(p,q)\le\sum_{i=1}^{k-1}\overline{\rho}(a_i,a_{i+1})+\overline{\rho}(p,q)+\sum_{j=1}^{k-1}\overline{\rho}(b_j,b_{j+1}),$$

where we have $a_1=p$, a_2, ..., $a_n=x$, $y=b_1$, b_2, ..., $b_m=q$. Thus,

$$v=\overline{\rho}(x,y)\ge\underline{\rho}(p,q)-\sum_{i=1}^{k-1}\overline{\rho}(a_i,a_{i+1})-\sum_{j=1}^{k-1}\overline{\rho}(b_j,b_{j+1}).$$

For each p and q, these inequalities are equivalent to v being larger than or equal to the largest of the right-hand sides, i.e., the difference between $\underline{\rho}(p,q)$ and the smallest possible values of the sums $\sum\limits_{i=1}^{k-1}\overline{\rho}(a_i,a_{i+1})$ and $\sum\limits_{j=1}^{k-1}\overline{\rho}(b_j,b_{j+1})$. By definition of the function $\rho_M(a,b)$, these largest values are equal to $\rho_M(p,x)$ and $\rho_M(y,q)$. Thus, for each p and q, all the corresponding inequalities are equivalent to

$$v\ge\underline{\rho}(p,q)-\rho_M(p,x)-\rho_M(y,q).$$

All these inequalities are equivalent to being larger than or equal to the largest of these differences, i.e., to $v\ge\rho_m(x,y)$.

The proposition is proven.

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