What Is the Most Robust Certainty Measure for Interval-Valued Degrees*

Vladik Kreinovich[1], Chenyi Hu[2], Makenzie Spurling[2],
Huixin Zhan[3], and Victor S. Sheng[3]

[1]Department of Computer Science, University of Texas at El Paso, El Paso, TX 79968, USA
vladik@utep.edu

[2]Department of Computer Science, University of Central Arkansas, Conway, AR 72035-0001, USA
CHu@uca.edu,mspurling1@cub.uca.edu

[3]Department of Computer Science, Texas Tech University, Lubbock, TX 79409-3104, USA
huixin.zhan@ttu.edu,victor.sheng@ttu.edu

Abstract

People are often not 100% confident in their opinions. In the computers, absolute confidence is usually described by 1, absolute confidence in the opposite statement by 0. It is therefore reasonable to estimate intermediate degree of confidence by a number between 0 and 1. Many people cannot indicate an exact number corresponding to their degree of confidence, so a natural idea is to allow them to mark an interval of possible values. Since each person is not fully confident, a natural way to get a more definite conclusion is to combine opinions of several people. In this combination, it is reasonable to take the opinions of more certain people with higher weight. For this purpose, we need to assign, to each subinterval of the interval $[0, 1]$, a measure of certainty, so that intervals $[0, 0]$ and $[1, 1]$ – corresponding to absolute certainty – will get measure 1 and the interval $[0, 1]$ corresponding to absolute uncertainty will get measure 0. People have similar trouble marking the exact endpoints of the interval-valued degree; it is therefore reasonable to select a certainty measure that will be the least sensitive – i.e., the most robust – to inevitable small changes in these endpoints. In this paper, we find the most robust certainty measure. It turns out to be exactly the measure that has been shown to be empirically successful in analyzing people's opinions.

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1 Formulation of the Problem

Need for degrees. In many real-life situations, information about the world comes from human opinion – e.g., from expert opinion. In the ideal case, we ask a "yes"-"no" question to an appropriate person – a medical doctor, a professional engineer, a police officer – and we get a "yes" or "no" answer. In many cases, however, this person is not 100% sure what to answer. In such situations, a reasonable idea is to ask this person to mark his/her degree of confidence by a number of the scale of 0 to 1, 1 meaning absolutely sure, 0 meaning absolutely sure that the answer is "no"; see, e.g., [1, 2, 3, 4, 5, 7]. Alternatively, we can ask to mark a degree of confidence, e.g., by a number from 0 to 10, and then divide the corresponding value by 10.

Need for interval-valued degrees. Some people can describe their degree of confidence by an exact number, but for many people, distinguishing between, e.g., degree 0.7 and degree 0.71 is difficult. For such people, it is more convenient to mark their degree of confidence by an interval of possible values, e.g., $[0.7, 0.8]$. This way, people who have no information about the question can mark their degree of confidence by the whole interval $[0, 1]$ of possible values, people who have a little bit of confidence can mark it as $[0.1, 1]$, etc.

Need for a certainty measure. When we do not get a fully confident answer from a single person, a natural idea is to ask several people and combine their answers. More certain answers should get more weight in such a combination, less certain answers should get less weight, and an answer $[0, 1]$ – meaning that the person has no opinion at all about the given question – should get weight 0. We therefore need to have a numerical measure of certainty that describes, based on the interval $[a, b]$ provided by a person, to what extent this person's opinion should be taken into account.

A heuristic certainty measure. To combine opinions, [6] proposed the following heuristic idea. An interval $[a, b] \subseteq [0, 1]$ can be described by its center $m = \frac{a+b}{2} \in [0, 1]$ and its radius (half-width) $r = \frac{b-a}{2} \in [0, 0.5]$. Absolute uncertainty means that the expert's degree of confidence is equal to $[0, 1]$. In this case, both the midpoint and the radius are equal to 0.5. So, the further away the midpoint from 0.5 and the further away the radius from 0.5, the larger the certainty. The corresponding two distances are equal to $|0.5 - m|$ and $|0.5 - r|$, so a natural measure of certainty is the sum

$$h(a, b) = |0.5 - m| + |0.5 - r|. \tag{1}$$

Question. The formula (1) helps provide a reasonable combination of people's opinions [6]. However, since this formula is purely heuristic, a natural question is: can this empirical formula be derived based on some fundamental ideas? or maybe the attempts of such a derivation will lead to even better performing formulas?

What we do in this paper. In this paper, we show that the formula (1) can indeed be derived from fundamental principles – and thus, its empirical success is not accidental.

2 Idea, Definitions and the Main Result

Let us first simplify the expression $h(a,b)$. To be able to compare our results with the empirical formula (1), let us first simplify this formula. The radius r is always smaller than or equal to 0.5, so $0.5 \ge r$, and we can replace $|0.5 - r|$ with $0.5 - r$. Depending on the sign of the difference $0.5 - m$, we get the following expressions:

- If $0.5 - m = 0.5 - \dfrac{a+b}{2} \ge 0$, i.e., equivalently, if $a \le 1 - b$, then the expression (1) takes the form
$$0.5 - \frac{a+b}{2} + 0.5 - \frac{b-a}{2} = 1 - b.$$
- If $0.5 - m = 0.5 - \dfrac{a+b}{2} \le 0$, i.e., equivalently, if $1 - b \le a$, then the expression (1) takes the form
$$\frac{a+b}{2} - 0.5 + 0.5 - \frac{b-a}{2} = a.$$

In both cases, the expression (1) takes the form $h(a,b) = \max(a, 1-b)$.

Idea. Similarly to the fact that people cannot easily provide the exact value of the degree of confidence, they may have difficulty providing exact values of the interval endpoints a and b. For example:

- If I am estimating my degree of confidence on a 0-to-10 scale, I can mark it by $[8,9]$, which corresponds to the interval $[0.8, 0.9] \subseteq [0,1]$.
- However, if I was asked to estimate the same degree of confidence on a 0-to-9 scale, I may mark the corresponding degrees 0.8 and 0.9 by integers $9 \cdot 0.8 \approx 7$ and $9 \cdot 0.9 \approx 8$, as $[7,8]$, which corresponds to a slightly different interval $[0.77\ldots, 0.88\ldots] \subseteq [0,1]$.

In general, instead of the original interval $[a,b]$, the same person can express his/her degree of confidence by a nearby interval $[a', b']$, where a' is close to a and b' is close to b, i.e., where $|a - a'| \le \varepsilon$ and $|b - b'| \le \varepsilon$ for some small $\varepsilon > 0$ – i.e., equivalently, where $\max(|a - a'|, |b - b'|) \le \varepsilon$.

It is therefore desirable to come up with a certainty measure $c(a,b)$ which is the least sensitive (i.e., most robust) with respect to such a change. Let us describe this idea in precise terms.

Definition 1. *By a* certainty measure*, we mean a function $c(a,b)$ that assigns, to each interval $[a,b] \subseteq [0,1]$, a number $c(a,b) \in [0,1]$ for which $c(0,1) = 0$ and $c(0,0) = c(1,1) = 1$.*

Definition 2. *We say that a certainty measure is* the most robust *if for some K and for all a, b, a', and b', it satisfies the inequality*
$$|c(a,b) - c(a',b')| \le K \cdot \max(|a - a'|, |b - b'|), \qquad (2)$$
and the value K is the smallest possible.

Proposition. *The most robust certainty measure is $c(a,b) = \max(a, 1-b)$.*

Proof. For $a = a' = b = 0$ and $b' = 1$, the inequality (2) implies that $K \ge 1$. It is easy to check that the certainty measure (1) satisfies the inequality (2) with $K = 1$. So, to complete the proof, it is sufficient to show that the measure (1) is the only certainty measure that satisfies the inequality (2) with $K = 1$, i.e., the inequality

$$|c(a, b) - c(a', b')| \le \max(|a - a'|, |b - b'|). \quad (3)$$

Indeed, for $a' = 0$ and $b' = 1$, the inequality (3) implies that

$$c(a, b) \le \max(a, 1 - b). \quad (4)$$

For $a' = b' = 0$, we get $1 - c(a, b) \le \max(a, b)$. Since $a \le b$, we have $\max(a, b) = b$ and thus, $1 - c(a, b) \le b$ and

$$c(a, b) \ge 1 - b. \quad (5)$$

Finally, for $a' = b' = 1$, we get $1 - c(a, b) \le \max(1 - a, 1 - b)$. Since $a \le b$, we have $\max(1 - a, 1 - b) = 1 - a$ and thus, $1 - c(a, b) \le 1 - a$ and

$$c(a, b) \ge a. \quad (6)$$

Inequalities (5) and (6) imply that

$$c(a, b) \ge \max(a, 1 - b). \quad (7)$$

Inequalities (4) and (7) imply that $c(a, b) = \max(a, 1 - b)$.

The proposition is proven.

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