# On Consistency of Multi-Intervals: A Comment*

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**Abstract**

After each measurement, we get a set of possible values of the measured quantity. This set is usually an interval, but sometimes it is a union of several disjoint intervals – i.e., a multi-interval. The results of measuring the same quantity are consistent if the corresponding sets intersect. It is known that for any family of intervals, if every two intervals from this family have a non-empty intersection, then the whole family has a non-empty intersection. We use a known result from combinatorial convexity to show that for multi-intervals, even if we require that every $k$ multi-intervals from a given family have a common element, this still does not necessarily imply that the whole family is consistent.

**Keywords:** Multi-interval, consistency, combinatorial convexity
**AMS subject classifications:** 65G20, 65G30, 65G40, 52A35, 52A37

## 1 Formulation of the Problem

**Ubiquity of intervals and multi-intervals.** In practice, we rarely know the exact values of physical quantities. Our information about these values comes from measurements, and measurement are never absolutely accurate. As a result, all we know about the actual value $x$ after the measurement(s) is a set $X$ that contains this value.

In many practical cases, this set is an interval $[\underline{x}, \overline{x}]$, but sometimes, it is a union of several intervals – which is known as a *multi-interval.* A union of no more than $d$ intervals is called a *$d$-interval.*

*Terminological comment.* The above definition reflects the current use of the term "$d$-interval"; see, e.g., [1]. It should be mentioned that in older papers – e.g., in the paper [3] that we cite later – such unions are called *homogeneous $d$-intervals*, and the term "$d$-interval" means something else.

**Need to analyze consistency.** Sometimes, the measuring instrument malfunctions and we get what is called an outlier – which, in our case, means a set that does not

*Submitted: December 29, 2021; Revised: ?; Accepted: ?.

actually contain the actual value. In the absence of outliers, if we perform several measurements of the same quantity, then the resulting sets $X_1, \ldots, X_n$ have a non-empty intersection – since they all contain the actual value of the measured quantity.

**Case of intervals.** When all the sets $X_i$ are intervals, then to conclude that these intervals have a non-empty intersection, it is sufficient to check that every two intervals have a non-empty intersection.

Indeed, if every two intervals $[\underline{x}_i, \overline{x}_i]$ and $[\underline{x}_j, \overline{x}_j]$ have a common element $x$, then we have $\underline{x}_i \le x \le \overline{x}_j$ and thus, $\underline{x}_i \le \overline{x}_j$ for all $i$ and $j$. Hence, for $m \stackrel{\text{def}}{=} \max\limits_i \underline{x}_i$, we have $m \le \overline{x}_j$ for all $j$. Clearly, also $\underline{x}_i \le m$ for all $m$. Thus, we have $\underline{x}_i \le m \le \overline{x}_i$ for all $i$. So $m$ is indeed a common point of all the given intervals.

**What about multi-intervals?** A natural question is: is a similar property true for $d$-intervals? If every two $d$-intervals from a family have a non-empty intersection, does the whole family have a non-empty intersection? If this is not true, maybe for some $k > 2$, if every $k$ $d$-intervals from the family have a non-empty intersection, then the whole family has a non-empty intersection?

It turns out that this is not true, no matter what $k$ we choose.

# 2 Main Result

**Proposition.** *For every $d \ge 2$ and for every $k \ge 2$, there exists a family $F$ of $d$-intervals such that every $k$ of them has a common point, but the whole family does not have a non-empty intersection.*

**Proof.** This result immediately follows from the main theorem of [3] (see also Chapter 30 of [1]). To be more precise, Part (i) of Theorem 1.1 from [3] states (if reformulated in our terms), that there exists a constant $c > 0$ such that for every $d \ge 2$ and $k \ge 2$, there exists a family $F$ of $d$-intervals for which:

- every subfamily of $k$ $d$-intervals has a non-empty intersection,
- but the smallest size of a set that intersects with all $d$-intervals from $F$ is greater than or equal to $c \cdot \dfrac{d^2}{\log(d)} \cdot (k-1)$.

For sufficiently large $k$, this lower bound is larger than 1, which means that we cannot have a single-point set that intersects with all $d$-intervals from the family $F$. This means exactly that the intersection of the whole family $F$ is empty.

*Example.* Here is an example – largely borrowed from [2] and [1] – of a family 2-intervals for which every two have a non-empty intersection but the intersection of all three of them is empty:

$$X_1 = [0,3] \cup [8,9]; \quad X_2 = [0,1] \cup [4,7]; \quad X_3 = [4,5] \cup [8,11];$$

$$X_4 = [0,3] \cup [10,11]; \quad X_5 = [2,3] \cup [4,7], \quad X_6 = [6,7] \cup [8,11].$$

Here:

$$X_1 \cap X_2 = [0,1], \quad X_1 \cap X_3 = [8,9], \quad X_1 \cap X_4 = [0,3],$$

$$X_1 \cap X_5 = [2,3], \quad X_1 \cap X_6 = [8,9],$$

$$X_2 \cap X_3 = [4,5], \quad X_2 \cap X_4 = [0,1], X_2 \cap X_5 = [4,7], \quad X_2 \cap X_6 = [6,7],$$

$$X_3 \cap X_4 = [10, 11], \quad X_3 \cap X_5 = [4, 5], \quad X_3 \cap X_6 = [8, 11],$$
$$X_4 \cap X_5 = [2, 3], \quad X_4 \cap X_6 = [10, 11],$$
$$X_5 \cap X_6 = [6, 7].$$

# Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes), and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

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