How General Is Fuzzy Decision Making?

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Abstract

In many practical situations, users describe their preferences in imprecise (fuzzy) terms. In such situations, fuzzy techniques are a natural way to describe these preferences in precise terms.

Of course, this description is only an approximation to the ideal decision making that a person would perform if we took time to elicit his/her exact preferences. How accurate is this approximation? When can fuzzy decision making – potentially – describe the exact decision making, and when there is a limit to the accuracy of fuzzy approximations?

In this paper, we show that decision making can be precisely described in fuzzy terms if and only if different parameters describing the alternatives are independent – in the sense that if for two alternatives, all other parameters are the same, then the preference between these two alternatives depends only on the differing values and doe not depend on the values of all other parameters.

Keywords: Fuzzy decision making, Traditional decision making, Utility

1 Formulation of the Problem

Fuzzy decision making: a brief reminder. Most of the time, when people make decisions, they do not formulate their criteria in precise terms. Instead, they formulate them by using imprecise (fuzzy) words from natural language.

For example, when you ask a person looking for a house what exactly he or she wants, this person will probably reply that he/she wants a house which is:

- located in a good neighborhood,
- reasonably large,
- not too expensive,
- not far away from the stores and entertainment district, etc.

All these terms – *good* neighborhood, *reasonably* large, etc. are imprecise.

A natural way to describe these criteria in precise terms is to use fuzzy techniques – techniques designed by Lotfi Zadeh specifically for translating words from natural language into precise, computer-understandable terms; see, e.g., [1, 4, 8, 12, 13, 15].

According to these techniques:

- First, for each criteria i, we design a *membership function* $\mu_i(x_i)$ that describes, for each possible house x, the degree – on the scale from 0 to 1 – to which the characteristic x_i corresponding to this criterion – cost, size, etc. – satisfies the i-th criterion.
- Once we have the degrees $\mu_1(x_1), \ldots, \mu_n(x_n)$ corresponding to all n criteria, we then use an appropriate "and"-operation (t-norm) $f_\&(a,b)$ to estimate the degree $\mu(x)$ to which all n criteria are satisfied as

$$\mu(x) = f_\&(\mu_1(x_1), \ldots, \mu_n(x_n)). \quad (1)$$

- After that, a reasonable idea is to select the alternative x for which this overall degree of satisfaction $\mu(x)$ is the largest possible.

This procedure is, of course, an approximation to the ideal exact decision making. Of course, every time we use imprecise words, what we get is an approximate description – in the case of decision making, it is an approximate description of our preferences.

There is a whole science of decision making that described how to elicit exact preferences and make exact decisions – we will recall its main ideas in the next section. Going ahead, according to decision theory, we should maximize the expected value of a special function called *utility* $u(x)$ that describes our preferences.

How accurately does fuzzy decision making approximate the exact one? A natural question is: how accurately does fuzzy decision making approximate the exact one? When can the exact decision making approximated by fuzzy one with any given accuracy, and when it cannot be thus approximated?

This is what we study in this paper. As a result of this study, we provide a clear answer to this question.

2 Decision Theory: A Brief Reminder

What we do in this section. In this section, we provide a brief description of the traditional decision theory; for details, see, e.g., [2, 3, 6, 7, 9, 10, 14].

Utility: the main notion of decision theory. Decision theory describes our preferences in precise numerical terms. Its main idea lies in the fact that if we select two alternatives:

- a very bad alternative A_- which is clearly worse than anything that we will actually encounter, and
- a very good alternative A_+ which is clearly better than anything that we will actually encounter,

then we can have a natural numerical scale by considering lotteries $L(p)$ in which we get:

- the very good alternative A_+ with probability p, and
- the very bad alternative A_- with the remaining probability $1-p$.

Clearly, the larger the probability of the very good alternative A_+, the better the lottery.

For each actual alternative A, if the probability p is small, then the lottery $L(p)$ is close to A_- and is, thus, worse than A; we will denote it by $L(p) < A$.

On the other hand, if the probability p is close to 1, then the lottery $L(p)$ is close to A_+ and is, thus, better than A: $A < L(p)$.

If we assume that the person has exact preferences, i.e., he/she can always decide which of the two alternatives is better for him/her, then for each p, we have either $L(p) < A$ or $A < L(p)$ (or $A \sim L(p)$ – meaning that to this person, the alternative A and the lottery $L(p)$ have equal value). Under this assumption, there is a threshold value u of the probability such that:

- for each probability p for which $p < u$, we have
$$L(p) < A, \text{ and}$$
- for each probability p for which $u < p$, we have
$$A < L(p).$$

This threshold value is known as the *utility* of the alternative A.

By definition of the utility, for each $\varepsilon > 0$, we have

$$L(u-\varepsilon) < A < L(u+\varepsilon).$$

For very small ε, we do not feel the difference between lotteries corresponding to probabilities u, $u-\varepsilon$, and $u+\varepsilon$. Thus, we can say that from the practical viewpoint, the alternative A is actually equivalent to the lottery $L(u)$. We will denote this by $A \equiv L(u)$.

Which alternative should we select? As we have mentioned, each alternative A_i is equivalent to the lottery $L(u_i)$, where u_i is the utility of this alternative. Thus, comparing alternatives is equivalent to comparing the corresponding lotteries $L(u_i)$.

We have also mentioned that when we compare several lotteries $L(p)$, then the larger the probability of the very good alternative A_+, the better. Thus, we have to select the alternative A_i for which the utility value u_i is the largest.

Why expected utility. One of the consequences of the above definition of utility is that in the case of uncertainty, we need to maximize the expected utility. Some folks – who are not very familiar with decision theory – mistakenly think that the maximization of expected utility is na additional (and not-well-justified) postulate, but it is not, it is a consequence of utility's definition.

Indeed, suppose that, as a result of some action a, we get:

- an alternative A_1 with probability p_1,
- an alternative A_2 with probability p_2,
- $\ldots$, and
- an alternative A_m with probability p_m.

What is the utility of this action? As we have mentioned, each alternative A_i is equivalent to a lottery in which we get:

- the very good alternative A_+ with probability equal to the utility u_i of this alternative, and
- the very bad alternative A_- with the remaining probability $1 - u_i$.

Thus, the whole action a is equivalent to the 2-stage lottery, in which:

- first, we select an alternative, so that each alternative A_i is selected with probability p_i, and
- then, depending on which alternative A_i we selected, we select A_+ with probability u_i and A_- with probability $1 - u_i$.

As a result of this 2-stage lottery, we get either A_+ or A_-. By considering all n possible ways to get A_+, in each of which we get A_+ with the probability $p_i \cdot u_i$, we conclude that the overall probability of selecting A_+ is equal to the sum of these values, i.e., to

$$u \stackrel{\text{def}}{=} p_1 \cdot u_1 + \ldots + p_m \cdot u_m. \quad (2)$$

Thus, the action is equivalent to the lottery in which we get A_+ with probability u and A_- with the remaining probability. By definition of utility, this means that the action has utility u.

And it so happens that the expression (2) – that describes this utility – is actually the expected value of the utility u_i. So, the principle of maximizing the expected utility indeed follows from the definition of utility.

3 So When Can Exact Decision Be Described in Fuzzy Terms?

The problem: reminder. Now that we recalled the traditional decision theory, let us go back to the original question.

In general decision theory, when we select an alternative x characterized by values $x_1, \ldots, x_n$, the recommendation is to select the alternative for which the utility $u(x) = u(x_1, \ldots, x_n)$ is the largest possible.

In fuzzy decisions, we select the alternative for which the expression (1) attains its largest possible value. When is such a representation possible for a given utility function – i.e., for the given preference relation?

Let us reformulate fuzzy decision making in utility terms. To be able to compare the two approaches, let us perform some reformulations. Specifically, we reformulate the fuzzy decision making in terms which are closer to utilities.

To do this, we can take into account that:

- while, in principle, there exist many different "and"-operations (t-norms),
- it is known (see, e.g., [11]) that for every t-norm $f_\&(a,b)$ and for every $\varepsilon > 0$, there exists an ε-close t-norm $g(a,b)$ of the type
$$g(a,b) = f^{-1}(f(a) \cdot f(b)), \quad (3)$$
for some strictly increasing function $f(x)$; here, $f^{-1}(x)$ denotes the inverse function.

Similarly to what we argued in the previous section, we can argue that for a sufficiently small ε, we cannot actually distinguish between ε-close degrees of confidence. Thus, from the practical viewpoint, we can safely assume that the t-norm actually has the form (3).

For such a t-norm, the formula (1) turns into

$$g(\mu_1(x_1), \ldots, \mu_n(x_n)) =$$
$$f^{-1}(f(\mu_1(x_1)) \cdot \ldots \cdot f(\mu_n(x_n))). \quad (4)$$

Since the function $f(x)$ is strictly increasing, maximizing the expression (4) is equivalent to maximizing the result of applying the function $f(x)$ to this value, i.e., to maximizing the expression

$$f(\mu_1(x_1)) \cdot \ldots \cdot f(\mu_n(x_n)). \quad (5)$$

Since logarithm is also a strictly increasing function, maximizing the product (5) is equivalent to maximizing its logarithm. Since the logarithm of the product is equal to the sum of the logarithms, the logarithm of the expression (5) can be described as the problem of maximizing the sum

$$u_1(x_1) + \ldots + u_n(x_n), \quad (6)$$

where we denoted $u_i(x_i) \stackrel{\text{def}}{=} \ln(f(\mu_i(x_i)))$.

Vice versa, if our decision problem can be described in the form (6), we can take, e.g., $f(x) = x$ (then $f_\&(a,b) = a \cdot b$), and $\mu_i(x_i) = C_i \cdot \exp(u_i(x_i))$, for some normalization constant C_i – to make sure that all the values do not exceed 1. One can easily see that for these "and"-operation and membership functions, maximizing the expression (1) is indeed equivalent to maximizing the formula (5).

Now the problem has been reformulated, so we can answer the original question. Now, the original problem – when decision making can be described in fuzzy terms – has been reformulated in precise terms:

When is a decision problem characterized by a utility function $u(x_1, \ldots, x_n)$ equivalent to maximizing the sum (5) of expressions describing the utilities of different parameters x_i characterizing the alternative?

Interestingly, this problem has already been solved in utility theory; see, e.g., [3, 5]. Namely, one can easily check that if we want to maximize the expression (5), then our preferences between different parameters are *independent* in the following sense: if we have two different situations differing only by the values of $x_i = x_i'$, then which alternative is better depends only on the relation between x_i and x_i' and does not depend on the values of the other parameters: if

$$(x_1, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n) <$$

$$(x_1, \ldots, x_{i-1}, x_i', x_{i+1}, \ldots, x_n)$$

for some values

$$x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n,$$

then for any other values

$$x_1', \ldots, x_{i-1}', x_{i+1}', \ldots, x_n',$$

we will have the similar relation:

$$(x_1', \ldots, x_{i-1}', x_i, x_{i+1}', \ldots, x_n') <$$

$$(x_1', \ldots, x_{i-1}', x_i', x_{i+1}', \ldots, x_n').$$

This is true not only when we have exact values of the parameters, the same property holds if we only know the probability distributions on the set of parameters.

It has been proven that this independence property uniquely characterizes the possibility of representation (5); namely:

- if the above independence property holds,
- then the maximized utility can be represented as the sum (5) of the terms $u_i(x_i)$ each depending only on the corresponding parameter.

So, we get the following answer to our questions.

4 Conclusion

A decision making problem, in which we compare alternatives x characterized by several parameters $x_1, \ldots, x_n$, can be represented in the equivalent form (1) corresponding to fuzzy decision making if and only if these parameters are independent – in the formal sense described in the previous section.

Crudely speaking, it means that when for two alternatives, all other alternatives are the same, our preference between these two alternatives deoends only on the values of the differing parameter and does not depend on the values of all other parameters.

- If this independence condition is satisfied, then fuzzy decision making can approximate the actual decision making as accurately as we want – in can even exactly represent the actual decision making.
- On the other hand, if the independence condition is not satisfied, then there is a limit on how accurately fuzzy decision making can approximate the actual decision making.

Acknowledgement

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes).

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478.

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