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Why Daubechies wavelets are so successful

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Abstract. In many applications, including analysis of seismic signals, Daubechies wavelets perform much better than other families of wavelets. In this paper, we provide a possible theoretical explanation for the empirical success of Daubechies wavelets. Specifically, we show that these wavelets are optimal with respect to any optimality criterion that satisfies the natural properties of scale- and shift-invariance.

Keywords: Daubechais wavelets, seismology, invariance

1. Formulation of the problem

Need for 1-D wavelets. The values of most physical quantities q change with time t: $q = q(t)$. In some cases, e.g., in celestial mechanics, we know the general shape of this dependence, i.e., we know that the function $f(t, c_1, \ldots, c_n)$ such that the actual dynamics $q(t)$ is determined by this expression $q(t) = f(t, c_1, \ldots, c_n)$ for some values of the parameters $c_1, \ldots, c_n$. In such cases:

- first, we use the values $q(t_1), \ldots, q(t_k)$ observed during some observation period to determine the values of these parameters, i.e., to solve the system of equations

$$q(t_i) = f(t_i, c_1, \ldots, c_n), \;\; i = 1, \ldots, k, \quad (1)$$

  with the unknowns c_i;
- then, we use the resulting values c_i to predict the to predict the future values $q(t)$ of the quantity q as $q(t) = f(t, c_1, \ldots, c_n)$.

Sometimes, the dependence on the parameters c_i is non-linear – so this system of equations is not easy to solve. However, if this is how q depends on time, there is nothing we can do about it.

In many other cases, however, we do not know the shape of the dependence. In such cases, it is also desirable to come up with a general formula $f(t, c_1, \ldots, c_n)$ – with not-too-many parameters c_i – that would adequately describe the dynamics of the quantity of interest. In such situations, it is reasonable to select a family for which the corresponding system of equations (1) is the easiest to solve – i.e., is a system of linear equations. For this purpose, we select a family for which the dependence $f(t, c_1, \ldots, c_n)$ is linear in terms of the unknowns, i.e., for which

$$f(t, c_1, \ldots, c_n) =$$

$$f_0(t) + c_1 \cdot f_1(t) + \ldots + c_n \cdot f_n(t) \quad (2)$$

for some functions $f_i(t)$.

Such representations are indeed actively used in data processing. For example, for a smooth dependence $q(t)$, it is reasonable to approximate it by a polynomial – i.e., by the sum of the first few terms of its Taylor expansion. In this case, the functions $f_i(t)$ are monomials t^j corresponding to non-negative integers j. For

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a periodic process with a known period T, we can use sines and cosines $\sin(j \cdot \omega \cdot t)$ and $\cos(j \cdot \omega \cdot t)$ for non-negative integers j, where $\omega = 2\pi/T$, etc.

However, many physical processes – e.g., seismic processes – are neither smooth nor periodical: they consist of time-localized bursts of activity. To describe such processes, it makes sense to use similarly time-localized functions $f_i(t)$. Such functions are known as *wavelets*; see, e.g., [1,2,4,6].

From generic wavelets to Daubechies wavelets. One of the computational advantages of Fourier series – i.e., of representing the desired dependence as a linear combination of sines and cosines – is that all the functions $f_i(t)$ used in this approximation can be obtained from each other by scaling and shift, i.e., they all have the form $f_i(t) = f_0(a_i \cdot t_b$ for some values a_i and b_i, where $f_0(t) \stackrel{\text{def}}{=} \sin(\omega \cdot t)$.

It turns out that we can select wavelets that satisfy a similar property – namely, we select the basic function $\varphi(t)$ known as the *mother wavelet*, and by taking the functions $f_i(t)$ of the form $\varphi(2^j \cdot t - \ell)$, where $j \geq 0$ and ℓ are integers. (There are also similar functions generated by another related function, known as the *father wavelet*.) For the resulting functions $f_i(t)$ to be efficient in representing and processing data, the mother wavelet must satisfy a certain linear functional equation.

This functional equation has many different solutions. Empirically:

- wavelets corresponding to some solutions work better, while
- wavelet corresponding to other solutions of the functional equations do not work so well.

To select a single solution – and thus, to fix a family of wavelets – we need to impose additional restrictions on the function $\varphi(t)$. To make computations easier – and to preserve linearity of the corresponding system of equation – it makes sense to impose restrictions which are linear in terms of $\varphi(t)$. A general such linear restriction has the form

$$\int c_m(t) \cdot \varphi(t)\, dt = b_m, \quad m = 1, \ldots, M, \qquad (3)$$

for some function $c_m(t)$.

Once we know one solution $\varphi_0(t)$ to the system (3) of linear equations, we can have an even simpler system of linear equations for the difference $\Delta(t) \stackrel{\text{def}}{=} \varphi(t) - \varphi_0(t)$:

$$\int c_m(t) \cdot \Delta(t)\, dt = 0, \quad m = 1, \ldots, M. \qquad (4)$$

Of course, once the equality $\int c(t) \cdot \Delta(t)\, dt = 0$ holds for all M functions $c_1(t)$, …, $c_M(t)$, the same equality holds for all possible linear combinations

$$c(t) = s_1 \cdot c_1(t) + \ldots + s_M \cdot c_M(t)$$

of these functions, i.e., for the whole M-dimensional linear space L of functions generated by the functions $c_m(t)$. From this viewpoint, the condition (4) can be described as requiring that

$$\int c(t) \cdot \Delta(t)\, dt = 0$$

for all functions $c(t)$ from an M-dimensional linear space of functions. Thus, the selection of a specific wavelet family means selecting an appropriate linear space L.

Ingrid Daubechies proposed to use $c_m(t) = t^{m-1}$, i.e., equivalently, to take, as L, the linear space of all polynomials

$$c(t) = a_0 + a_1 \cdot t + \ldots + a_{M-1} \cdot t^{M-1}$$

of order less than M; see, e.g., [2]. The resulting wavelets are known as *Daubechies wavelets.*

Empirical fact. In many practical applications – in particular, in processing seismic signals – Daubechies wavelets work very well, much better than many other wavelet families; see, e.g., [3] and references therein.

What we do in this paper. In this paper, we provide a possible theoretical explanation for the empirical success of Daubechies wavelets.

Specifically, we show that these wavelets are optimal with respect to any optimality criterion that satisfies the natural properties of scale- and shift-invariance.

2. Analysis of the problem

We need to select an M-dimensional linear space. As we have mentioned earlier, selecting a family of wavelets is equivalent to selecting an M-dimensional linear space L of functions. In these terms, the question is:

What is the optimal selection of an M-dimensional linear space of functions?

We will only consider smooth (differentiable) functions $c(t)$. In wavelet analysis, the corresponding functions $c(t)$ are differentiable. In view of this, in this paper, we will also limit ourselves to the case when all the functions $a(t)$ from the linear space L are differentiable.

Comment. This diffentiability requirement makes sense: e.g., it is known that every continuous function $c(t)$ can be approximated, with any given accuracy, by smooth functions. Since from the practical viewpoint, a very small difference is not noticeable, it thus makes sense to assume that all the functions $c(t)$ are differentiable.

However, a reader should be warned that it is not possible to follow this argument too far: Actually, some wavelets are *not* smooth. Even Daubechies wavelets of higher order M, while smooth, are not infinitely differentiable: if we differentiate the corresponding mother wavelet again and again, we will eventually reach a function which is not differentiable at some points.

What does "optimal" mean. Usually, when we say that an alternative $A_{\rm opt}$ is optimal, it means that:

- there is a numerical characteristic $F(A)$ describing the imperfection of different alternatives, and
- the alternative $A_{\rm opt}$ has the smallest value of this characteristic.

For example, for different wavelet families A, we can take, e.g., the mean squared accuracy $F(A)$ with which the use of the first few wavelets from this family approximates signals from the given set of signals.

However, this is not the only way to describe optimality. For example, in the above example, we may have several different families with the same smallest possible value of the mean squared accuracy. In such a case, we can use this non-uniqueness to minimize some other characteristic $G(A)$: e.g., the average computation time needs to get the corresponding approximation. We then say that the alternative A is better or of the same quality as an alternative B – we will denote it by $A \le B$ – if:

- either $F(A) < F(B)$,
- or $F(A) = F(B)$ and $G(A) \le G(B)$.

If this additional numerical criterion does not lead to a unique selection of an alternative, we can minimize something else, etc., until we reach the *final* optimality criterion – for which there is exactly one optimal alternative.

No matter how complex our comparison, in all these cases, we have a relation $A \le B$ between the two alternative describing that A is better or of the same quality as B.

Of course, each alternative has the same quality as itself $A \le A$, and if $A \le B$ and $B \le C$, then $A \le C$. Thus, we arrive at the following definition.

Definition 1. *Let $\mathcal{A}$ be a set. Its elements will be called* alternatives.

- *By an* optimality criterion*, we mean a binary relation $\le$ on this set which satisfies the following two properties:*
  - *for every $A \in \mathcal{A}$, we have $A \le A$ (*reflexivity*), and*
  - *for all $A, B, C \in \mathcal{A}$, if $A \le B$ and $B \le C$, then $A \le C$.*
- *An alternative $A_{\rm opt}$ is called* optimal *with respect to the optimal criterion $\le$ if we have $A_{\rm opt} \le A$ for all $A \in \mathcal{A}$.*
- *An optimality criterion $\le$ is called* final *if for this criterion, there exists exactly one optimal alternative.*

Natural invariance properties. We are interested in describing how a quantity changes with time. We describe this dependence in numerical terms, as a dependence $q(t)$ of the numerical value of the quantity q on the numerical value of time t.

However, the numerical value of time depends on the selection of the measuring unit and on the selection of the starting point. If we replace the original unit with a new one which is a times smaller – e.g., consider seconds instead of minutes – then all numerical values of time are multiplied by a. The corresponding linear transformation $t \mapsto a \cdot t$ is known as *scaling*.

Similarly, if we replace the original starting point for measuring time with a new starting point which is t_0 moments earlier, then this value t_0 will be added to all numerical values of time. The corresponding linear transformation $t \mapsto t + t_0$ is known as *shift*.

In general, if we change both the unit and the starting point, we replace t with $a \cdot t + t_0$ – i.e., we get a linear transformation.

The numerical values change, but the physical process remains the same. From this viewpoint, it is reasonable to require that the relative quality of two different methods should not change if we simply change the unit and/or the starting point. In terms of linear spaces – that describe different wavelet families – we thus arrive at the following definition.

Definition 2.

- *By a* linear transformation*, we mean a function* $T(t) = a \cdot t + t_0$ *for some values* a *and* t_0.
- *For each linear transformation* T *and each function* $e(t)$*, by the* result $T(e)$ *of applying* T *to* e *we mean a function* $e(T(t))$.
- *For each* M*-dimensional linear space* L *of smooth functions, by the result* $T(L)$ *of applying* T *to* L *we mean the linear space formed by the functions* $T(e)$ *for* $e \in L$.
- *We say that the optimality criterion* $\le$ *on the set* $\mathcal{L}$ *of all* M*-dimensional linear spaces of smooth functions is* invariant *if for every two spaces,* $L_1 \le L_2$ *implies that* $T(L_1) \le T(L_2)$.

Now, we can formulate our main result.

3. Main result

Proposition. *For every final invariant optimality criterion on the set of all* M*-dimensional linear spaces of smooth functions, all elements of the optimal family* $L_{\rm opt}$ *are polynomials of order less than* M.

Comments.

- Thus, we have indeed proven that the linear space corresponding to Daubechies wavelets is optimal – and thus, so, in this sense, Daubechies wavelets are optimal.
- The following proof follows ideas first described in [5].

Proof. Let us first prove that the optimal family $L_{\rm opt}$ is itself invariant, i.e., that $T(L_{\rm opt}) = L_{\rm opt}$.

Indeed, the fact that $L_{\rm opt}$ is optimal means that $L_{\rm opt} \le L$ for all families L, in particular, for all families of the type $T^{-1}(L)$, where T^{-1} is the inverse transformation. So, $L_{\rm opt} \le T^{-1}(L)$ for each L. By using invariance of the optimality crterion, we conclude that $T(L_{\rm opt}) \le L$ for every L, i.e., that the linear space $T(L_{\rm opt})$ is also optimal. However, the optimality criterion $\le$ is final, which means that there is only one optimal space, so indeed, $T(L_{\rm opt}) = L_{\rm opt}$.

Let us now select any basis $e_1(t)$, ..., $e_M(t)$ in the optimal linear space. Invariance of the linear space $L_{\rm opt}$ means, in particular, that for each i and for each t_0, the shifted function $e_i(t + t_0)$ also belongs to this linear space, i.e., that

$$e_i(t + t_0) = \sum_{j=1}^{M} C_{ij}(t_0) \cdot e_j(t) \tag{5}$$

for some coefficients C_{ij} depending on t_0. If we select M different moments of time $t_1, \ldots, t_M$, we get a system of M linear equations to determine these coefficients $C_{ij}(t_0)$ in terms of the functions e_j:

$$e_i(t_1 + t_0) = \sum_{j=1}^{M} C_{ij}(t_0) \cdot e_j(t_1);$$

$$\ldots \tag{6}$$

$$e_i(t_M + t_0) = \sum_{j=1}^{M} C_{ij}(t_0) \cdot e_j(t_M).$$

In general, the solution of a system of linear equations is a linear combination of the left-hand sides. The left-hand sides $e_i(t_k + t_0)$ are differentiable functions of t_0, thus, all the coefficients $C_{ij}(t_0)$ are also differentiable. So, all the functions in the formula (5) are differentiable. Thus, we can differentiate both sides with respect to t_0, and get

$$e_i'(t + t_0) = \sum_{j=1}^{M} C_{ij}'(t_0) \cdot e_j(t). \tag{7}$$

In particular, for $t_0 = 0$, we get

$$e_i'(t) = \sum_{j=1}^{M} c_{ij} \cdot e_j(t), \tag{8}$$

where we denoted $c_{ij} \stackrel{\rm def}{=} C_{ij}'(0)$. So, for M functions $e_1(t), \ldots, e_M(t)$, we have a system of M linear differential equations with constant coefficients.

It is known that a general solution to such a system is a linear combination of expressions of the type $t^k \cdot \exp(\alpha \cdot t)$, where:

– the value α is an eigenvalue of the matrix $\|c_{ij}\|$, and
– the value k is an non-negative integer which is smaller that the multiplicity of this eigenvalue.

Similarly, another consequence of invariance is that for every i and for every a, the function $e_i(a \cdot t)$ also belongs to the optimal space $L_{\rm opt}$, i.e., that i.e., that

$$e_i(a \cdot t) = \sum_{j=1}^{M} C_{ij}(a) \cdot e_j(t) \tag{9}$$

for some coefficients C_{ij} depending on a. If we select M different moments of time $t_1, \ldots, t_M$, we get a system of M linear equations to determine these coefficients in terms of e_j:

$$e_i(a \cdot t_1) = \sum_{j=1}^{M} C_{ij}(a) \cdot e_j(t_1);$$

$$\ldots \tag{10}$$

$$e_i(a \cdot t_M) = \sum_{j=1}^{M} C_{ij}(a) \cdot e_j(t_M).$$

In general, the solution of a system of linear equations is a linear combination of the left-hand sides. The left-hand sides $e_i(a \cdot t_k)$ are differentiable functions of t_0, thus, all the dependence of all the coefficients $C_{ij}(a)$ is also differentiable. So, all the functions in the formula (9) are differentiable. Thus, we can differentiate both sides with respect to a, and get

$$t \cdot e_i'(a \cdot t) = \sum_{j=1}^{M} C_{ij}'(a) \cdot e_j(t). \tag{11}$$

In particular, for $a = 1$, we get

$$t \cdot e_i'(t) = \sum_{j=1}^{M} c_{ij} \cdot e_j(t), \tag{12}$$

where we denoted $c_{ij} \stackrel{\rm def}{=} C_{ij}'(1)$. Let us introduce an auxiliary variable $x \stackrel{\rm def}{=} \ln(t)$, so that $t = \exp(x)$ and $dx = dt/t$. Then,

$$t \cdot \frac{de_i}{dt} = \frac{de_i}{dt/t} = \frac{de_i}{dx},$$

so the formula (12) takes the form

$$\frac{dE_i(x)}{dx} = \sum_{j=1}^{M} c_{ij} \cdot E_j(x), \tag{13}$$

where we denoted $E_i(x) \stackrel{\rm def}{=} e_i(\exp(x))$. So, for M functions $E_1(x)$, …, $E_M(x)$, we also have a system of M linear differential equations with constant coefficients, and thus, each of these functions is a linear combination of the expressions of the type

$$x^k \cdot \exp(\alpha \cdot x).$$

Thus, each function $e_i(t) = E_i(\ln(x))$ is a linear combination of functions

$$(\ln t)^k \cdot \exp(\alpha \cdot \ln(t)) = (\ln t)^k \cdot t^\alpha. \tag{14}$$

One can check that the only way to have a function representable both as a linear combination of these expressions (14) and a linear combination of expressions $t^k \cdot \exp(\alpha \cdot t)$ is when in the formula (14), we have $k = 0$ and α is an integer, i.e., when each function $e_i(t)$ is a linear combination of monomials t^k – i.e., a polynomial.

To complete the proof, let us show that all polynomials can only have degree $< M$.

Indeed, suppose that the optimal linear space $L_{\rm opt}$ contains a polynomial of degree d, i.e., a function

$$e^{(0)}(t) = a_0 \cdot t^d + a_1 \cdot t^{d-1} + \ldots,$$

with $a_0 \neq 0$. The optimal linear space is invariant with respect to shift, so for each h, the function $e^{(0)}(t+h)$ also belong to this space. Since $L_{\rm opt}$ is a linear space, it also contains any linear combination of the two functions $e^{(0)}(t)$ and $e^{(0)}(t+h)$, in particular, their difference

$$e^{(1)}(t) \stackrel{\rm def}{=} e^{(0)}(t+1) - e^{(0)}(t). \tag{14}$$

One can check that this difference is a polynomial

$$e^{(1)}(t) = d \cdot a_0 \cdot t^{d-1} + \ldots \tag{15}$$

of degree $d-1$. By applying this difference again and again, we get a polynomial

$$e^{(2)}(t) = e^{(1)}(t+1) - e^{(1)}(t)$$

of degree $d-2$, etc., all the way to a polynomial

$$e^{(d)}(t) = e^{(d-1)}(t+1) - e^{(d-1)}(t)$$

of degree 0, i.e., to a constant.

These $d+1$ polynomials $e^{(0)}(t), \ldots, e^{(d)}(t)$ are all linearly independent: indeed, each linear combination

$$c_{i_1} \cdot e^{(i_1)}(t) + \ldots + c_{i_k} \cdot e^{(i_k)}(t)$$

for some $i_1 < i_2 < \ldots$ and all $c_{i_j} \neq 0$ starts with a non-zero term proportional to t^{d-i_1} and thus, cannot be identically 0.

According to linear algebra, in an M-dimensional space, we can have no more than M linearly independent elements, so here we have $d+1 \leq M$, thus $d \leq M-1$, hence indeed $d < M$.

The proposition is proven.

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