How the Pavement's Lifetime Depends on the Stress Level and on the Dry Density: An Explanation of Empirical Formulas

Edgar Daniel Rodriguez Velasquez, Vladik Kreinovich, Olga Kosheleva, and Hoang Phuong Nguyen

Abstract We show that natural invariance ideas explain the empirical dependence on the pavement's lifetime on the stress level and on the dry density.

1 First Problem: Dependence on Stress

General description of the phenomenon. Road pavements have a limited lifetime. As the vehicles pass over the pavement, eventually, the pavement develops fatigue cracking and needs to be repaired.

Clearly, the larger the load, the larger the stress σ, the larger the strain ε, and thus, the smaller the number of repetitions N before the fatigue cracking.

Empirical fact. To estimate the expected lifetime of the pavement, we need to know how the number of repetitions N to fatigue cracking depends on the stress σ, i.e., we need to know the following dependence:

$$N = f(\sigma). \tag{1}$$

Edgar Daniel Rodriguez Velasquez
Department of Civil Engineering, Universidad de Piura in Peru (UDEP), Av. Ramón Mugica 131 Piura, Peru, e-mail: edgar.rodriguez@udep.pe
and
Department of Civil Engineering, University of Texas at El Paso, 500 W. University El Paso, TX 79968, USA, e-mail: edrodriguezvelasquez@miners.utep.edu

Vladik Kreinovich and Olga Kosheleva
University of Texas at El Paso, El Paso, Texas 79968, USA
e-mail: vladik@utep.edu, olgak@utep.edu

Hoang Phuong Nguyen
Division Informatics, Math-Informatics Faculty, Thang Long University, Nghiem Xuan Yem Road Hoang Mai District, Hanoi, Vietnam, e-mail: nhphuong2008@gmail.com

Empirically, this dependence is described by the following formula; see, e.g., [2, 3, 5]:

$$N = N_0 \cdot \exp(-k \cdot \sigma). \qquad (2)$$

This formula applies both to the cases when we analyze how N depends on the stress measured at the top layer of the pavement and on the stress measured at the bottom layer of the pavement.

Natural question. How can we explain this empirical formula?

2 Analysis of the Problem

There are several components of stress. In both top-of-pavement and bottom-of-pavement situations, when we apply the formula (1), we measure the overall stress, and implicitly assume that all this stress is caused by the traffic. In reality, in addition to the traffic-related stress, there are other stresses that also contribute to the eventual deterioration of the pavement. Indeed, even an unused road eventually develops cracks, due, e.g., to weather-induced stress.

The empirical formula relates only to the traffic-related stress. The additional stress – e.g., weather-related stress – follows different cyclic patterns than the traffic-related stress: typically a yearly pattern, or – in case of the rain – a pattern corresponding to each instance of rain.

The number of repetitions of these additional stresses is therefore much smaller than the number of repetitions caused by the traffic passing over the pavement. Thus, in comparison with the number of traffic-related repetitions, the number of weather-related repetitions can be safely ignored.

Strictly speaking, only the traffic-related part σ_t of the stress – the part that repeats many times per day – affects the number of repetitions N. In other words, strictly speaking, we should look for a formula

$$N = f(\sigma_t). \qquad (3)$$

How can we estimate the traffic-related part of the stress. All we measure is the overall stress σ, which is equal to the sum $\sigma_t + \sigma_o$ of the traffic-related stress σ_t and the stress σ_o caused by other factors. So, to estimate the traffic-related stress σ_t, we need to subtract, from the measured stress σ, our estimate of the remaining stress σ_o.

Resulting uncertainty in estimating the traffic-related stress. The problem is that this remaining stress can only be estimated rather approximately. If instead of the original estimate σ_0 we used a slightly different estimate σ'_o, then, instead of the original estimate $\sigma_t = \sigma - \sigma_o$, we get a somewhat different estimate

$$\sigma'_t = \sigma - \sigma'_0 = (\sigma - \sigma_o) + (\sigma_o - \sigma'_o) = \sigma_t + \delta,$$

where we denoted $\delta \stackrel{\text{def}}{=} \sigma_o - \sigma'_o$.

As a result, the exact same situation can be described by two somewhat different values σ_t and $\sigma'_t = \sigma_t + \delta$.

3 Invariance Requirement

Natural invariance idea. Since the exact same situation can be described by two somewhat different values σ_t and $\sigma_t = \sigma'_t + \delta$, it is therefore reasonable to require that these two somewhat different values lead to the exact same predictions of the pavement lifetime.

Naive approach to invariance does not work. Of course, we cannot interpret this requitement literally, as saying that we should have $f(\sigma_t) = f(\sigma_t + \delta)$: if we impose this requirement for all σ_t and all δ, this would lead to a physically meaningless conclusion that the function $f(\sigma_t)$ is a constant, i.e., that the pavement's lifetime does not depend on the stress at all.

Solution: using experience of physics. However, the experience of physics shows that we do not need to take this requirement literally; see, e.g., [4, 6]. For example, we know that the formula $v = d/t$ describing the velocity v as a function of distance d and time t does not depend on what measuring unit we select for distance: we can describe the distance d in meters, or we can describe it in centimeters, resulting in a different numerical value $d' = 100 \cdot d$.

In this example, invariance does not mean that we always have $d/t = d'/t$: the formula $v = d/t$ does remain valid if we use a different unit for measuring distance, but for this formula to remain valid, we also need to correspondingly change the unit that we use for measuring velocity: in the above example, from m/sec to cm/sec. In this case, we get $v' = d'/t$, where v' is the numeric value of velocity in the new units.

From this viewpoint, a reasonable idea is to require that for $\sigma'_t = \sigma_t + \delta$, the dependence $N = f(\sigma_t)$ should lead to $N' = f(\sigma'_t)$, where N' is the description of the pavement lifetime in correspondingly different units.

But can we use the experience of physics in our case? At first glance, the physics-motivated idea does not seem to work in our case:

- in contrast to quantities like distance or velocity, where we do need to select a measuring unit to get a numerical value,
- the number of repetitions N is simply an integer, no measuring is required.

Yes, we can. However, a more detailed analysis shows that the situation is not that uniquely determined. Indeed, usually, when the vehicle goes over a pavement location:

- we can count it as a single stress cycle,
- or we can consider the situation more accurately and take into account that every time each wheel is passing over, it is a different cycle.

Thus, depending on how we count it, what was a single cycle in one counting becomes several cycles if we consider it differently.

In mathematical terms, this means that we can describe the same traffic history:

- by the number N and
- by a different number $N' = c \cdot N$, where c is the average number of axles per vehicle.

Different estimates of the number c can lead, in general, to different re-scalings of N'.

Resulting formulation of the invariance requirement. From this viewpoint, the above requirement that the dependence $N = f(\sigma_t)$ not change if we change our estimate for σ_o takes the following form.

For every real number δ,
there exists an appropriate value $c(\delta)$ – which depends on δ –
such that if $N = f(\delta_t)$, then we should have $N' = f(\sigma'_t)$,
where $\sigma'_t = \sigma_t + \delta$ and $N' = c(\delta) \cdot N$.

4 Resulting Explanation

Reduction to a functional equation. Substituting the expressions $\sigma'_t = \sigma_t + \delta$ and $N' = c(\delta) \cdot N$ into the formula $N' = f(\sigma'_t)$, we conclude that $c(\delta) \cdot N = f(\sigma_t + \delta)$. Since $N = f(\sigma_t)$, we thus conclude that

$$c(\delta) \cdot f(\sigma_t) = f(\sigma_t + \delta). \tag{3}$$

Resulting explanation. It is known – see, e.g., [1] – that every measurable solution to the functional equation (3) has the form $f(\sigma) = N_0 \cdot \exp(-k \cdot \sigma)$. Thus, we have indeed explained the empirical formula (2).

5 Second Problem: Dependence on Dry Density

Empirical formula. The formula (2) describes how the pavement lifetime depends on the stress provided that all other parameters remain constant. The corresponding values N and k depend on the dry density ρ of the underlying soil. This dry density is called *maximum dry density* since when the pavement is built, we try to maximize the dry density of the compacted soil. The dependence of the lifetime N on stress and dry density is described by the following formula [5]:

$$\ln(N) = k_4 \cdot \ln\left(\frac{\rho}{\omega}\right) \cdot \left(1 - \frac{\sigma}{k_5 \cdot UCS}\right), \tag{4}$$

for some parameters k_4, ω, k_5, and UCS, i.e., equivalently, by the formula:

$$\ln(N) = a_{00} + a_{01} \cdot \sigma + a_{10} \cdot \ln(\rho) + a_{11} \cdot \sigma \cdot \ln(\rho), \tag{5}$$

for some values a_{00}, a_{01}, a_{10}, and a_{11}.

How can we explain this empirical formula?

What we know already. We know, from the previous text, that for each fixed value ρ, the dependence of N on σ has the form (2), i.e., equivalently, that $\ln(N)$ is a linear function of σ:

$$\ln(N) = \ln(N_0) - k \cdot \sigma. \tag{6}$$

Let us analyze how N depends on the dry density ρ.

Corresponding invariance. We can use different units to measure dry density. If we replace the original unit with a new unit which is λ times smaller, then all numerical values get multiplied by λ, i.e., instead of each original numerical value ρ, we get a new numerical value $\rho' = \lambda \cdot \rho$ to describe the exact same physical situation.

There is no reason to prefer one specific measuring unit. It is therefore reasonable to require that the dependence of the lifetime N on the density ρ be the same, no matter what measuring unit we use. Similarly to the case when we analyzed the dependence of N on σ, it is reasonable to formulate this requirement in the following precise terms:

For every real number $\lambda > 0$,
there exists an appropriate value $c(\lambda)$ – which depends on λ –
such that if $N = f(\rho)$, then we should have $N' = f(\rho')$,
where $\rho' = \lambda \cdot \rho$ and $N' = c(\lambda) \cdot N$.

Reduction to a functional equation. Substituting the expressions $\rho' = \lambda \cdot \rho$ and $N' = c(\lambda) \cdot N$ into the formula $N' = f(\rho')$, we conclude that $c(\lambda) \cdot N = f(\lambda \cdot \rho)$. Since $N = f(\rho)$, we thus conclude that

$$c(\lambda) \cdot f(\rho) = f(\lambda \cdot \rho). \tag{7}$$

Resulting formula. It is known – see, e.g., [1] – that every measurable solution to the functional equation (7) has the form $f(\rho) = N_0 \cdot \rho^a$ for some values N_0 and a. Thus, $\ln(N) = a \cdot \ln(\rho) + \ln(N_0)$, i.e., $\ln(N)$ is a linear function of ρ. So:

- for each value ρ, the logarithm $\ln(N)$ of the lifetime $N = f(\sigma, \rho)$ is a linear function of σ, and
- for each value σ, the logarithm $\ln(N)$ is a linear function of $\ln(\rho)$.

Thus, the function $\ln(N)$ is a bilinear function of σ and $\ln(\rho)$, i.e., has the desired form (5). So, the empirical dependence (5) is also explained.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), and HRD-1834620 and HRD-2034030 (CAHSI Includes), and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

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