Why Best-Worst Method Works Well

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***Abstract*—In many cases, experts are much more accurate when they estimate the ratio of two quantities than when they estimate the actual values. For example, if it difficult to accurately estimate the height of a person on a photo, but if we have two people standing side by side, we can easily estimate to what extent one of them is taller than the other one. To get accurate estimates, it is therefore desirable to use such ratio estimates. Empirical analysis shows that to obtain the most accurate results, we need to compare all the objects with either the "best" object – i.e., the object with the largest value of the corresponding quantity – or the "worst" object – i.e., the object with the smallest value of this quantity. In this paper, we provide a theoretical explanation for this empirical observation.**

***Index Terms*—expert estimates, best-worst method, estimation accuracy**

I. Introduction

Formulation of the practical problem. In many application areas, we rely on human estimates of different quantities. For example, when police investigates a crime, they rely on witnesses' estimates of the suspect's height and/or weight. In general:

- we have n objects, and
- for each object i, $i = 1, \ldots, n$, we want to know the corresponding value a_i of a quantity a.

Estimates $\widetilde{a}_i$ of untrained people are usually not very accurate – and thus, not very helpful. What we humans are much better at is *comparing* different values. For example:

- if we see two people, especially if we see then side by side,
- then we can conclude that one of them is, e.g., 20% taller than the other.

Similarly:

- an instructor may not be able to accurately predict how exactly each student will perform on a test, but
- usually, instructors can predict who will do better and who will do worse, and how better and how worse.

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So, for some pairs (i, j), we ask the user to estimate the ratio

$$\frac{a_i}{a_j}$$

of the corresponding values. Based on these estimates, we want to reconstruct the values of the desired quantities.

Practical limitation. In general, the more information we have, the more accurate the resulting estimate. From this viewpoint:

- the more questions we ask about different pairs,
- the better.

However, as the number of objects increases, the number of pairs increases quadratically, as

$$\frac{n \cdot (n-1)}{2} \sim n^2.$$

For large n, it becomes un-realistic to ask questions about all the pairs. With such possibility in mind, it is necessary to ask the smallest possible number of questions. A natural idea is:

- to select one of the objects i_0, and
- to only ask for ratios between this object and all other objects.

Empirical fact. It has been shown that we can get the most accurate estimates:

- either if we compare all the quantities with the smallest one,
- or if the compare all the quantities with the largest one;

see, e.g., [4]. This is known as the *best-worst method*.

What we do in this paper. In this paper, we provide a theoretical explanation for this empirical result.

II. Let Use Formulate This Problem in Precise Terms

What we mean by reconstructing the values a_i. In order to formulate the problem in precise terms, let us first clarify what we mean by reconstructing the values a_i.

Of course, if we only know the ratios, we cannot uniquely determine the actual values. Indeed:

- if we multiply all the values a_i by the same constant c,
- then the ratios remain the same, while
- the numerical values change.

To avoid this non-uniqueness, a natural idea is to select some object i_0 for which we simply take $a_{i_0}^{\rm new} \stackrel{\rm def}{=} \widetilde{a}_{i_0}$.

This means, in effect, that we replace the original measuring unit with a new one, which is

$$\frac{\widetilde{a}_{i_0}}{a_{i_0}}$$

times smaller than the original measuring unit. In terms of this unit, the new values $a_i^{\rm new}$ of the desired quantity take the form

$$a_i^{\rm new} = a_i \cdot \frac{\widetilde{a}_{i_0}}{a_{i_0}}.$$

Since we multiply all the values of the quantity by the same constant

$$c = \frac{a_{i_0}^{\rm new}}{a_{i_0}},$$

the ratios remain the same:

$$\frac{a_i^{\rm new}}{a_j^{\rm new}} = \frac{a_i}{a_j}.$$

A natural question. A natural question is: which object i_0 should we select?

Once we selected i_0, how can we reconstruct the values a_i? If our estimates of the ratios were exact, then, in principle, by comparing all the objects with the selected object i_0, we could get the exact values of all the quantities a_i:

- either as

$$a_i = \frac{a_i}{a_{i_0}} \cdot a_{i_0}$$

- or alternatively, as

$$a_i = \left(\frac{a_{i_0}}{a_i}\right)^{-1} \cdot a_{i_0}.$$

In practice, we do not know the exact ratios

$$\frac{a_i}{a_j},$$

we only know the estimates w_{ij} for these ratios:

$$w_{ij} \approx \frac{a_i}{a_j}.$$

So, by using these estimates instead of the actual ratios, we can provide estimates $\widetilde{a}_i$ for the desired quantity by using:

- either the formula $\widetilde{a}_i = \widetilde{a}_{i_0} \cdot w_{ii_0}$,
- or, alternatively, the formula $\widetilde{a}_i = \widetilde{a}_{i_0} \cdot w_{i_0 i}^{-1}$.

But is there a difference between these two approaches? At first glance, it may look like it does not matter what method we use, since the estimated ratios

$$frac{a_i}{a_{i_0}} \text{ and } \frac{a_{i_0}}{a_i}$$

are simply inverses to each other, so a consistent person should select estimates which are inverses as well, i.e., estimates for which

$$w_{ii_0} = \frac{1}{w_{i_0 i}}.$$

However, it is well known that people are not perfectly consistent (see, e.g., [2]). So, in general, these two estimates will lead to results:

- which are exactly mutually reverse and
- which, thus, may lead to different estimates for the values a_i of the desired quantity.

Need to take uncertainty into account. In practice, as we have mentioned, we can only estimate the ratios with some accuracy. Let us denote the accuracy with which we estimate the ratios by ε:

- This can be the mean square value of the difference between the actual ratio

$$\frac{a_i}{a_j}$$

  and our estimate w_{ij}.
- This can also be the largest possible absolute value of this difference

$$\left|\frac{a_i}{a_j} - w_{ij}\right|.$$

How shall we compare different selections. Since the ratios are only known with some inaccuracy, the resulting estimates of a_i are also inaccurate, i.e., they contain, in general, approximation error. In this paper, we will use two ways to compare the accuracy of different approaches:

- by comparing the worst-case approximation error and
- by comparing the mean square approximation error;

see, e.g., [3], [5].

Now, we are ready to formulate the corresponding problem in precise terms.

III. Precise Formulation of the Problem and the Resulting Solution: Case When Experts Estimate the Ratios a_i/a_{i_0}

Description of the case. Let us first consider the case when we ask experts to provide estimates w_{ii_0} for the ratios

$$\frac{a_i}{a_{i_0}}.$$

What is the approximation error of estimating a_i. In this case, we estimate a_i as $w_{ii_0} \cdot \widetilde{a}_{i_0}$. We have denoted the accuracy of estimating the ratio w_{ii_0} by ε. Let us analyze how this affect the accuracy of estimating a_i.

For this purpose, let us denote the approximation error of approximating any quantity x with its approximate value $\widetilde{x}$ by $\Delta x \stackrel{\rm def}{=} \widetilde{x} - x$. For a_i, the exact value – in the new measuring unit – is

$$a_i = \frac{a_i}{a_{i_0}} \cdot \widetilde{a}_{i_0},$$

while our estimate of this value is equal to $\widetilde{a}_i = w_{ii_0} \cdot \widetilde{a}_{i_0}$. Thus, the approximation error $\Delta a_i \stackrel{\rm def}{=} \widetilde{a}_i - a_i$ is equal to

$$\Delta a_i = \left(w_{ii_0} - \frac{a_i}{a_{i_0}}\right) \cdot \widetilde{a}_{i_0} = \Delta w_{ii_0} \cdot \widetilde{a}_{i_0},$$

where we denoted

$$\Delta w_{ii_0} \stackrel{\rm def}{=} w_{ii_0} - \frac{a_i}{a_{i_0}}.$$

So, the desired approximation error Δa_i of estimating a_i is obtained from the approximation error Δw_{ii_0} of estimating the corresponding ratio by multiplying it by $\widetilde{a}_{i_0}$. Thus, whether we talk about the accuracy as the mean square approximation error or the largest possible approximation error, the accuracy δ_i with which we estimate a_i can be obtained from the accuracy ε of estimating w_{ii_0} by multiplying it by the same the same number $\widetilde{a}_{i_0}$ (see [3], [5]):

$$\delta_i = \varepsilon \cdot \widetilde{a}_{i_0}.$$

Worst-case approach. In the worst-case approach, we minimize the worst-case approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\rm def}{=} \max_{i\ne i_0} \delta_i = \varepsilon \cdot \widetilde{a}_{i_0}.$$

Thus, to minimize this approximation error, we need to select, as the reference object i_0, the object with the *smallest* possible value of a_i. This explains one of the choices that turned out to be empirically successful.

Mean-square approach. In the mean-square approach, we minimize the mean-square approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\rm def}{=} \sqrt{\frac{1}{n-1}\cdot \sum_{i\ne i_0} \delta_i^2} = \sqrt{\frac{1}{n-1}\cdot \sum_{i\ne i_0} (\varepsilon \cdot \widetilde{a}_{i_0})^2} = \varepsilon \cdot \widetilde{a}_{i_0}.$$

This is the exact same expression as in the worst-case approach. So, to minimize this approximation error, we also need to select, as the reference object i_0, the object with the *smallest* possible value of a_i – which is exactly one of the choices that turned out to be empirically successful.

IV. Precise Formulation of the Problem and the Resulting Solution: Case When Experts Estimate the Ratios a_{i_0}/a_i

Description of the case. Let us now consider the case when we ask experts to provide estimates w_{i_0i} for the ratios

$$\frac{a_{i_0}}{a_i}.$$

What is the approximation error of estimating a_i. In this case, we estimate a_i as $w_{ii_0}^{-1}\cdot \widetilde{a}_{i_0}$. We have denoted the accuracy of estimating the ratio w_{ii_0} by ε. Let us analyze how this affect the accuracy of estimating a_i.

In general, suppose that we approximate a quantity x by a value $\widetilde{x}$, with approximation error $\Delta x \stackrel{\rm def}{=} \widetilde{x} - x$, we have $x = \widetilde{x} - \Delta x$, and then we use this estimate to estimate the value $y = f(x)$ for s given function $f(x)$. In this case, our estimate $\widetilde{y}$ for y is obtained by plugging in the approximate value $\widetilde{x}$ into the formula $y = f(x)$, i.e., $\widetilde{y} = f(\widetilde{x})$. Thus, the approximation error Δy of estimating y is equal to

$$\Delta y = \widetilde{y} - y = f(\widetilde{x}) - f(x) = f(\widetilde{x}) - f(\widetilde{x} - \Delta x).$$

Approximation errors are usually small, so that terms which are quadratic or higher order in terms of these errors can be safely ignored; see, e.g., [1], [6]. For example, for the accuracy of 20%, the square is 4% which is much smaller. So, we expand the right-hand side of the above expression for Δy in Taylor series and safely ignore quadratic and higher order terms – leaving only linear terms in this expansion. As a result, we get

$$\Delta y = f'(\widetilde{x}) \cdot \Delta x,$$

where $f'(x)$, as usual, means the derivative.

Whether we look for the largest possible absolute value of Δy or for its mean-square value, this value can be obtained by multiplying the accuracy of approximating x by $|f'(x)|$; see, e.g., [3], [5].

In our case, we have $x = w_{i_0i}$ and $f(x) = x^{-1} \cdot \widetilde{a}_{i_0}$, thus $f'(x) = -x^{-2} \cdot \widetilde{a}_{i_0}$. So, the accuracy δ_i with which we approximate a_i is equal to

$$\delta_i = w_{i_0i}^{-2} \cdot \widetilde{a}_{i_0} \cdot \varepsilon.$$

Here,

$$w_{i_0i} \approx \frac{a_{i_0}}{a_i} \approx \frac{\widetilde{a}_{i_0}}{\widetilde{a}_i},$$

so

$$w_{i_0i}^{-2} \approx \left(\frac{\widetilde{a}_{i_0}}{\widetilde{a}_i}\right)^{-2} = \frac{(\widetilde{a}_i)^2}{(\widetilde{a}_{i_0})^2},$$

and thus,

$$\delta_i \approx \frac{(\widetilde{a}_i)^2}{(\widetilde{a}_{i_0})^2} \cdot \widetilde{a}_{i_0} \cdot \varepsilon = \frac{(\widetilde{a}_i)^2}{\widetilde{a}_{i_0}} \cdot \varepsilon = (\widetilde{a}_i)^2 \cdot \frac{1}{\widetilde{a}_{i_0}} \cdot \varepsilon.$$

Worst-case approach. In the worst-case approach, we minimize the worst-case approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\rm def}{=} \max_i \delta_i = \max_{i\ne i_0} (\widetilde{a}_i)^2 \cdot \frac{1}{\widetilde{a}_{i_0}} \cdot \varepsilon.$$

Thus, to minimize this approximation error, we need to select, as the reference object i_0, the object with the *largest* possible value of a_i. This explains another of the two choices that turned out to be empirically successful.

Mean-square approach. In the mean-square approach, we minimize the mean-square approximation error, i.e., we minimize the quantity

$$\delta \stackrel{\rm def}{=} \sqrt{\frac{1}{n-1}\cdot \sum_{i\ne i_0} (\widetilde{a}_i)^4} \cdot \frac{1}{\widetilde{a}_{i_0}} \cdot \varepsilon.$$

To minimize this approximation error, we also need to select, as the reference object i_0, the object with the *largest* possible value of a_i – which is exactly one of the choices that turned out to be empirically successful.

V. Conclusions

To accurate estimate the values of a quantity based on expert estimates, it is important to take into account that experts estimate the ratios of different values much more accurately than the values themselves. It is therefore advisable to select one object, and to ask the expert to compare all other objects with the selected one.

Empirical analysis shows that to achieve the best accuracy, we should select, as the reference object, either the "best" object – i.e., the object with the largest value of the quantity of interest – or the "worst" object, i.e., the object with the smallest value of this quantity. In this paper, we have provided a theoretical explanation for this empirical fact.

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