Causality: Hypergraphs, Matter of Degree, Foundations of Cosmology

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Abstract The notion of causality is very important in many applications areas. Because of this importance, there are several formalizations of this notion in physics and in AI. Most of these definitions describe causality as a crisp ("yes"-"no") relation between two events or two processes – cause and effect. However, such descriptions do not fully capture the intuitive idea of causality: first, often, several conditions are needed to be present for an effect to occur, and, second, the effect is often a matter of degree. In this paper, we show how to modify the current description of causality so as to take both these phenomena into account – in particular, by extending the notion of directed acyclic graph to hypergraphs. As a somewhat unexpected side effect of our analysis, we get a natural explanation of why, in contrast to space-time of Special Relativity – in which division into space and time depends on the observer, in cosmological solutions there is a clear absolute separation between space and time.

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1 Formulation of the Problem

Causality is an important notion in many areas of research, in many practical situations. To be able to solve the corresponding practical problems, it is important to have an adequate description of what is causality.

Because this notion is so important, there are many formal descriptions of causality, in physics (see, e.g., [2, 4, 5, 10, 11]), in Artificial Intelligence (AI) – see, e.g., [9] – and in many other areas. The very fact that there are many different description is a good indication that the current definitions are not perfect – and most authors writing about causality explicitly state that their definitions, while useful in many practical cases, do not fully capture the intuitive idea of causality.

One of the important aspects of causality that it not properly captured by the current definitions is that in some situations, causality is a matter of degree. This has been stated, on many occasions, by Lotfi Zadeh and other researchers working in fuzzy techniques; see, e.g., [1, 3, 6, 7, 8, 12]; this has been shown on a physical level; see, e.g., [4, 5]. In this paper, we show how we can extend the current descriptions of causality to take this – and other – important aspects of causality into account.

2 What Is Causality: A Brief Reminder and the Need to Distinguish Between Two Aspects of Causality

Two main areas where the notion of causality is used. In other to motivate the following definitions, let us recall what we usually mean by causality. This notion is very important both:

- in everyday life – and, as an extension, in Artificial Intelligence (AI), and
- in physics.

Let us briefly describe how this notion is used in both application areas. Once we come up with this description, we will emphasize the fact that is not as widely known as it should be – that while in these two areas, we use the same term "causality", in these two areas, we talk about two *different* aspects of this notion. Thus, a proper formalization of this notion must take this difference into account.

Commonsense and AI aspect of causality: an example. If a road pavement needs repairs while it was supposed to still last, we need to know what caused it:

- it could be that there were mistakes in design,
- it could be that the design was correct, but the wrong materials were used, or, more generally, that some mistakes were made when this road segment was built;
- it could be that there were unusual extreme weather conditions that were not predicted and that caused the premature pavement deterioration;
- it could be that the load on this road segment was much higher than anticipated, and this caused the pavement deterioration;

- it could be that the formulas describing how pavement deteriorates with time – formulas that were used in the original design – were incorrect, etc.

In this sense, when we say that a caused b, this means that if – everything else being equal – we would not have a, then we would not see b either. For example, if we conclude that the use of a wrong asphalt mix caused the pavement deterioration, this means that if proper material was used, the road would not have deteriorated – and that in a similar situation, if we use the proper material, the road will not deteriorate before its expected lifetime.

Causality in physics: an example. The notion of causality is very important in physics too, but there, it means something else. Specifically, in physics, when we say that an event e causally affect an event e', this means that a change in *e can* make some change in e'.

What is the difference between these two aspects. If a soccer player kicks a ball and it goes into the opponent's gate, then, both from the commonsense viewpoint and from the physics viewpoint, this kick was the cause of the resulting goal. So far so good, it may look like these two notions are the same, but here is a counterexample. In real life, there is usually some wind, and that wind slightly affects the trajectory of the ball.

- From the viewpoint of common sense, the wind was clearly *not* the cause of the goal – unless it was such a strong gust of wind that the ball (that was originally supposed to miss) was blown into the gate.
- However, from the viewpoint of theoretical physics, the wind *did* have a causal effect on the ball's trajectory – in contrast, e.g., to the wind 1000 km away at this same moment that did not have any way to causally affect the ball's trajectory.

In short, in the first approximation:

- in common sense, a causes b if whenever we have a, we have b as well, i.e., when the presence of a determines the presence of b;
- in theoretical physics, a causally affects b if a has some effect on b – but does not necessarily uniquely determines all the aspects of b.

Towards a more realistic description. The above first-approximation description implicitly assumed that everything is deterministic. In real life – and in physics – we can rarely fully predict the future events. At best, we can predict the *probabilities* of different future events; see, e.g., [2, 11]. With this in mind, the proper description of the two aspects of causality takes the following form:

- in common sense, a causes b if *a uniquely determines* the probability distribution of possible values of the quantities characterizing the event b, and
- in theoretical physics, a causally effects b if a has *some effect* on the probability distribution of possible values of the quantities characterizing the event b.

Comment. Of course, these explanations are still approximate: in reality, in common sense, we take into account that some minor changes may still happen, so we can have slightly different probability distributions of b – but, from the practical viewpoint, this description provides a good description of real-life situations; see, e.g., [9].

3 How Causality Is Usually Described – and What Is Missing in This Usual Description

How causality is usually described. The usual description of causality is based on the following two natural properties of causality:

- if a causally influences b, then b cannot casually influence a – unless, of course, a and b are the same (and we extend the definition of causality to this trivial case);
- if a causally influences b and b causally influences c, then a casually influences c.

It makes sense, by the way, to formally extend the notion "a causally influences b" to the case when $a = b$ – since, as we have mentioned earlier, this notion manes that a change in a can cause a change in b – and, of course, a change in a causes a change in a.

The same two properties are true if instead of "causally influences", we use "causes". In both cases, if we denote "a causally influences b" or "a causes b" by $a \le b$, and if we extend the definition to allow the event to casually influence itself, we get the following properties for all a, b, and c:

- $a \le a$
- if $a \le b$ and $b \le a$, then $a = b$; and
- if $a \le b$ and $b \le c$, then $a \le c$ (*transitivity*).

In mathematics, a binary relation with these properties is known as an *order relation* – sometimes called *partial order*, to distinguish it from the case of *linear* (*total*) order, when for every a and b, we have either $a \le b$ or $b \le a$.

Because of this, causality is usually described as a *partial* order.

The first (known) limitation of the usual description. In practice, sometimes, several phenomena have to be present to cause the observed effect. For example, a building may be designed so as to withstand an earthquake, and it can be designed to withstand a tsunami, but if both unexpectedly happen at about the same time – as happened in Japan in 2011 – the building may collapse. In this case, the cause of the collapse is not just the earthquake – an earthquake itself would not have caused the collapse, and not the tsunami, but the fact that both were present.

A more mundane example is that to open a bank vault (or even to open a safe deposit box stored in a bank), you need two keys – none of them would open the vault by itself. It is difficult to describe this phenomenon by an order relation between the events.

A similar phenomenon can be imagined in physics, where it is important to describe cases in which an event a cannot affect an event e' unless some other event e'' was present.

The second (known) limitation of the usual description. The second known limitation of the current definition of causality as a partial order is that it does not take into account that causality is a matter of degree: in some cases, a strongly influences b, in other cases, it only affects b with some degree. As a result, when we a long causal chain in which a_1 affects a_2, a_2 affects a_3, etc., until a_{n-1} affects a_n, we can formally say that a_1 affects a_n – it is usually expressed by saying that the flapping of a butterfly's wings affects a future storm on another continent – but in reality, the degree of this effect decreases at every link of this chain, so that at the very end, the effect is not practically noticeable.

What we do in the following sections. In the following sections, we show how these two limitations can be overcome, i.e., how we can generalize the usual definition to take these two features of causality into account.

4 How to Take into Account that We May Need Several Events to Cause the Observed Phenomenon

Towards a natural generalization of the order relation. In the deterministic approximation to the traditional commonsense description, the relation $a \le b$ means that if we know the state of a, we can uniquely predict the state of b. In the case that we want to describe, knowing the state of a is not enough to uniquely predict the state of b – to make this prediction, we need to known the state of several events $a_1, \ldots, a_n$. In other words, here, instead of knowing the state of a single event a, we need to know the state of all the events from some set A. It is therefore reasonable to consider the corresponding relation $A \le b$ that means that once we know the states of all the events from the set A, then we can uniquely determine the state of the event b.

Of course, the way we described it means that if we add other elements to the set A, we would still get this relation $A \le b$. So, a natural idea to distinguish "true causality" – which we will denote by $A < b$ – from the above relation is to define it as $A \le b$ (i.e., A influences b) but b is not an element of the set A and no proper subset of A influences the event b. So, if $A < b$, then we cannot have $A' < b$ for any proper subset $A' \subset A$, $A' \ne A$.

Comment. In particular, when A is the empty set, we can have $\emptyset < b$, meaning that all the properties of the event b are pre-determined, nothing can affect them.

What is the natural analogue of avoiding the casuality loop, when a causes b and b causes a. The original definition os causality avoid non-intuitive loops by requiring that if $a \le b$ and $b \le a$, then $a = b$. In our case, a proper description is that if $A < b$, then we cannot have $B < a$ for any element $a \in A$ and for any set B including b.

What is the natural analogue of transitivity. In the traditional transitivity, if b causes c, and a causes b, then a causes c. What if, in our case, we have $B < c$? Then, to be able to conclude that $A < c$, we need to make sure that all events $b \in B$ can be predicted by A, i.e., that $A < b$ for all $b \in B$. In this case, the set A uniquely determine c – but maybe not all elements of A are needed to predict c, i.e., all we can conclude is that $A \le c$, i.e., that some subset of the set A causes c.

Resulting definition. Thus, we arrive at the following definition.

Definition 1. *Let U be a set. By a* causality relation *on U, we mean a relation $< \subseteq 2^U \times U$ between subsets of U and elements of U for which the following properties are satisfied:*

- *if $A < b$, then $b \notin A$;*
- *if $A < b$, $A' \subset A$ and $A' \ne A$, then $A' \not< b$;*
- *if $A < b$, then for any $a \in A$ and for any set B containing b, we cannot have $B < a$;*
- *if $A < b$ for all $b \in B$ and $B < c$, then we have $A' < c$ for some subset $A' \subseteq A$ of the set A.*

Terminological comment. Partial order can be described by a directed acyclic graph (DAG), in which objects are vertices, and pairs (a,b) for which $a \le b$ and $a \ne b$ are edges. In this section, we consider a generalization of this notion to the case when the relation $A \le b$ involves, in general, more than one element $a \in A$ and thus, more than two objects. Such a generalization of a graph – in which we consider subsets consisting of more than 2 elements – is known as a *hypergraph*. From this viewpoint, the above definition provides a generalization of the notion of a directed acyclic graph to hypergraphs.

An alterative description. We started this section with the “determination” relation $A \le b$ meaning that knowing A uniquely determines b. In terms of the causality relation $A < b$, this means that either $b \in A$ or there exists a subset $A' \subset A$ for which $A' < b$.

A somewhat inconvenient issue with this notion is that, in contrast to the original definition of causality $a \le b$ that relates two objects of the same type, the relation $A \le b$ related objects of different types: sets and elements. It is therefore natural to extend this relation to relation between two objects of the same type, namely, between sets: we can defined $A \le B$ if knowing all information about A provides us with the ability to predict all the values of all the elements $b \in B$. This possibility means that we can predict the values of each state $b \in B$, i.e., that we have $A \le b$ for all $b \in B$.

This new relation between sets has the following natural properties:

- if $B \subseteq A$, then $A \le B$;
- if $A \le B$ and $A \subseteq A'$, then $A' \le B$;
- if $A \le B$ and $B' \subseteq B$, then $A \le B'$.

In terms of the new relation, transitivity takes a very straightforward form: if $A \le B$ and $B \le C$, then $A \le C$.

Other properties of the causality relation can also be described in terms of this relation $A \le B$ if we take into account:

- that $A \le b$ simply means $A \le \{b\}$, and
- that, as we have mentioned, $A < b$ means that $b \notin A$ and that for all proper subsets $A' \subset A$, we have $A' \not< b$.

5 How to Take into Account that Causality is a Matter of Degree

Why it is necessary to take this into account. Another limitation of the traditional approach to causality is that this approach is black-and-white (crisp): in this approach, either an event a can influence event b or it cannot. In practice, at each moment of time, due to imperfection of measuring instruments, we can only detect effects of certain size – i.e., we can only confirm that certain pairs (a,b) for which a can influence b. As technology progresses, we can detect weaker and weaker effects – and thus, confirm that other pairs are also causally connected.

A natural way to describe this situation is to assign, to each pair of events (a,b), a degree $d(a,b)$ to which a can influence b: the larger the effect, the larger this degree. At each moment of time, we can measure the effect with a certain accuracy and thus, we know this degree with the similar accuracy. A natural way to describe such uncertainty is by saying the we know the approximate value $\widetilde{d}$ of this degree, and we know the accuracy δ, i.e., that the only thing we know about the actual (unknown) degree d is that this degree if located somewhere in the interval $[\widetilde{d}-\delta,\widetilde{d}+\delta]$.

What are the consequences of taking this into account. When we consider causality as crisp, then transitivity can be use indefinitely: if a_1 causes a_2, and a_2 causes a_2, etc., and a_{n-1} causes a_n, then a_1 causes a_n. In practice, the effect decreases with each transition: a_1 has a strong effect on a_2, a_2 may also have a strong direct effect on a_3, but the resulting indirect effect of a_1 on a_3 is smaller – since every time we have an effect, there is also some random influence that interferes with this effect. After many steps, the original effect becomes practically non-observable – just like if we transmit a message many times though noisy channels, the message will become corrupted more and more and eventually, will become completely unrecognizable.

In precise terms, instead of crisp transitivity, it is natural to expect the degree-based (fuzzy) version, when or each a, b, and c, the degree $d(a,c)$ to which a affects c is bounded by an "and"-operation (t-norm) $f_\&$ applied to the degrees $d(a,b)$ and $d(b,c)$:

$$d(a,c) \ge f_\&(d(a,b),d(b,c)).$$

This lead to a more realistic picture of causality. For example, if we use a t-norm for which we can have $f_\&(a,b) = 0$ for some $a > 0$ and $b > 0$, then eventually, the degree of influence can get to 0. Thus, we can have, for example, a closed time loop, in which going to the future brings us back to the past – provided that the path was sufficiently long; see, e.g., [10]. This possibility makes physical sense: in billions of years, all traces of the original event will disappear.

Interesting consequence of this idea. The idea of causality as a matter of degree does not just provide us with a better fit with physical intuition. In this section, we will show that this idea also – somewhat surprisingly – helps us to understand a certain fact from fundamental physics, namely, a somewhat counterintuitive transition between special relativity theory (and local effects of generic relativity) – that describe the local space-time, and cosmology that describes the global structure of space-time. Let us explain what is counterintuitive in this transition.

One of the main principles of special relativity theory is the relativity principle, according to which, by observations inside the system, there is no way to determine whether the system is at rest or moving with a constant speed in the same direction. This principle was first formulated by Galileo who noticed that when a ship is moving in a calm sea, then, if we are inside a cabin with no windows, we cannot tell whether the ship is moving or not. Einstein combined this principle with the empirical fact that the speed of light has the same value for all observers – and concluded that a priori, there is no fixed time coordinate: time (and corresponding notion of simultaneity) differ for different observers. Neither time interval Δt nor spatial distance ρ are invariant: they change from observer to observer, The only quantity that remains invariant is the so-called *proper time*, which, in distance units, takes the form $\tau = \sqrt{c^2 \cdot (\Delta t)^2 - \rho^2}$. The impossibility to separate time and space is the main reason why in special relativity, we talk about 4-dimensional space-time, and its division into space and time depends on the observer. Transformation between coordinate systems corresponding to different observers is described by so-called Lorenz transformations.

In contrast, in most cosmological models, there is a very clearly determined time coordinate, there is a clear separation into space and time. Why is that? Why cannot we have cosmology that has the same symmetries as local space-time?

When we consider the usual – crisp – causality, the only answer to this question is that observations support cosmological models in which time and space are separated, and support equations of General Relativity that lead to these models. In this description, there is nothing fundamental about these types of cosmologies – from the purely mathematical viewpoint, there is nothing wrong with considering space-time of special relativity.

Interestingly, the situation changes if we take into account that causality is a matter of degree. It turns out that if we make this assumption, then we cannot keep all the symmetries of special relativity – cosmological split between time and space becomes inevitable. Let us describe this result in precise terms.

To come up with this explanation, let us recall that in Special Relativity, the speed of light c is the largest possible speed, so an event $e = (t, x_1, x_2, x_3)$ can influence the event $e' = (t', x'_1, x'_2, x'_3)$ if and only if we can get from e to e' by traveling iwth a speed not exceeding the speed of light, i.e., if and only if the distance between the corresponding spatial points (x_1, x_2, x_3) and $x'_1, x'_2, x'_3)$ is smaller than or equal to the distance $c \cdot (t' - t)$ that someone traveling with the speed of light can cover during the time interval $t' - t$.

Definition 2.

- *Let c be a positive constant; we will call it* speed of light.
- *By* space-time of special relativity, *we mean an ordered set $(M,\le)$ in which $M = \mathbb{R}^4$ is the set of all 4-D tuples $e=(t,x_1,x_2,x_3)$ of real numbers, and the ordering relation has the form*

$$e=(t,x_1,x_2,x_2)\le e'=(t',x_1',x_2',x_3')\Leftrightarrow$$

$$c\cdot(t'-t)\ge\sqrt{(x_1-x_1')^2+(x_2-x_2')^2+(x_3-x_3')^2}.$$

- *If $e=(t,x_1,x_2,x_2)\le e'=(t',x_1',x_2',x_3')$, then by the* proper time $\tau(e,e')$ *we mean the quantity*

$$\tau(e,e')=\sqrt{c^2\cdot(t'-t)^2-(x_1-x_1')^2-(x_2-x_2')^2-(x_3-x_3')^2}.$$

- *We say that a mapping $f:M\to M$ is a* symmetry *if it preserves causality relation and preserves proper time, i.e., if $e\le e'$ is equivalent to $f(e)\le f(e')$ and $\tau(e,e')=\tau(f(e),f(e'))$ for all events e and e' for which $e\le e'$.*

Comment. It is known that for every two pairs (e,e') and (g,g') for which $\tau(e,e')=\tau(g,g')$, there exists a symmetry f that transforms e into g and e' into g'.

Let us now describe what we mean by taking into account that causality is a matter of degree. We will call the corresponding descriptions realistic causality functions. Their definition uses the notion of an "and"-operation, so let us first define this auxiliary notion.

Definition 2. *By an* "and"-operation, *we mean a continuous function $f_\&:[0,1]\times[0,1]\to[0,1]$ for which $f_\&(1,1)=1$.*

Comments.

- Every t-norm – as defined in fuzzy logic – satisfies this condition. However, this definition is much weaker than the usual definition of t-norm in fuzzy logic: it allows many functions which are not t-norms – e.g., they are no necessarily associative. The reason for us formulating such a weak definition is that we want to prove the main result of this section under most general assumptions. For example, our result stands if we consider non-commutative and non-associative operations instead of t-norms – as long as they satisfy the above definition.
- Continuity at the point $(1,1)$ means that for every $\varepsilon>0$ there exists $\delta>0$ such that if $x\ge 1-\delta$ and $y\ge 1-\delta$, then $f(x,y)\ge 1-\varepsilon$.

We want to define a function $d(e,e')$ that describes, for each two events e and e' for which $e\le e'$, the degree to which e can influence e'. What are the natural property of such a function?

- Of course, it is a fact that each event e "causes" itself – in the sense that if we know the event e, we can uniquely describe this same event. So, we must have $d(e,e)=1$.

- The fact that causality is a matter of degree means that the case when $e = e'$ should be the only case when we have $d(e,e') = 1$.In all other cases, we must have $d(e,e') < 1$.
- In physics, most dependencies are continuous, so it is reasonable to require that the function $d(e,e')$ should also be continuous.
- Finally, for the case when $e \le e' \le e''$, we should have

$$d(e,e'') \ge f_\&(d(e,e'),d(e',e'')).$$

Thus, we arrive at the following definition.

Definition 3. *Let $f_\&$ be an "and"-operation. By a* realistic causality function *on $(M,\le)$, we mean a continuous function $d(e,e')$ with values from the interval $[0,1]$ that is defined for all pairs (e,e') for which $e \le e'$ and that satisfies the following properties:*

- *$d(e,e') = 1$ if and only if $e = e'$, and*
- *$d(e,e'') \ge f_\&(d(e,e'),d(e',e''))$ for all $e,e',e'' \in M$.*

Definition 4. *We say that a realistic causality function is* invariant *with respect to a symmetry f if $d(e,e') = d(f(e), f(e'))$ for all $e \le'$.*

Proposition 1. *No realistic causality function is invariant with respect to all the symmetries.*

Comment. This result explains that, when we take into account that causality is a matter of degree, then we cannot have a space-time that has the same invariance properties as the space-time of Special Relativity: some corresponding symmetries have to be abandoned – and this explains why none of the current cosmological models has all these symmetries.

Proof. Let us prove this statement by contradiction. Let us assume that a realistic causality function d is invariant with respect to all the symmetries. Let us denote, for all $t \ge 0$, $e(t) \stackrel{\text{def}}{=} (t,0,0,0)$. We want to prove that in this case, we will have $d(e(0),e(1)) = 1$. This will contradict to the first part of the definition of the realistic causality function, according to which $d(e,e') = 1$ is also possible when $e = e'$. To prove the equality $d(e(0),e(1)) = 1$, we will prove that for every $\varepsilon > 0$, we have $d(e(0),e(1)) \ge 1-\varepsilon$.

According to the second comment after the definition of an "and"-operation, for this $\varepsilon > 0$, there exists a number $\delta > 0$ for which, if $x \ge 1-\delta$ and $y \ge 1-\delta$, then $f_\&(x,y) \ge 1-\varepsilon$. Thus, if we have an event e for which $d(e(0),e) \ge 1-\delta$ and $d(e,e(1)) \ge \delta$, then we have $f_\&(d(e(0),e),d(e,e(1))) \ge 1-\varepsilon$ and thus, since $d(e(0),e(1)) \ge f_\&(d(e(0),e),d(e,e(1)))$, that $d(e(0),e(1)) \ge 1-\varepsilon$.

To find such e, let us recall that, as we have mentioned, every two pairs (e,e') and (g,g') for which the proper time is the same can be transformed into each other by an appropriate symmetry f: $f(e) = g$ and $f(e') = g'$. Since the realistic causality function is invariant with respect to all the symmetries, this means that $d(e,e') = d(f(e), f(e')) = d(g,g')$. In other words, if two pairs have the same proper time, they have the same degree of causality. In mathematical terms, this means that the realistic causality function is a function of proper time, i.e., that $d(e,e') = F(\tau(e,e'))$ for some function $F(x)$.

In particular, for $e(t) = (t,0,0,0)$ with $t \ge 0$, we have $\tau(e(0),e(t)) = c \cdot t$. So, for these pairs, we have $d(e(0),e(t)) = F(c \cdot t)$. The function d is continuous and $e(t) \to e(0)$ as $t \to 0$. In the limit $t = 0$, we get $d(e(0),e(0)) = 1$. Thus, the function $F(x)$ is also continuous, tending to $F(0) = 1$ as $x \to 0$. By definition of the limit, this means that for every $\delta > 0$, there exists an $\nu > 0$ such that if $x \le \nu$, then $F(x) \ge 1 - \delta$.

Let us take $e = (t/2, c \cdot t/2 - \alpha, 0, 0)$. Here, as one can easily check, $e(0) \le e \le e(1)$, $\tau(e(0),e) = \tau(e,e(1))$ and the common value of proper time tends to 0 as $\alpha \to 0$. Thus, for sufficiently small α, we have $\tau(e(0),e) \le \nu$. Thus, we have $d(e(0),e) = F(\tau(e,e(0)) \ge 1 - \delta$ and similarly $d(e,e(1)) \ge 1 - \delta$. We have already shown that these two inequalities imply that $d(e(0),e(1)) \ge 1 - \varepsilon$. Since this is true for every $\varepsilon > 0$, this means that $d(e(0),e(1)) = 1$ – which contradicts to the definition of the realistic causality function. This contradiction proves that our assumption was wrong, and thus, indeed, no realistic causality function can be invariant with respect to all the symmetries.

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