The World Is Cognizable: An Argument Based on Hörmander's Theorem

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Abstract Is the world cognizable? Is it, in principle, possible to predict the future state of the world based on the measurements and observations performed in a local area – e.g., in the Solar system? In this paper, we use general physicists' principles and a mathematical theorem about partial differential equations to show that such prediction is indeed, theoretically possible.

1 Physical Equations and Physical Fields Are, in General, Smooth

Pre-quantum physics was based on the idea of determinism: if we know the values of all the physical quantities at some moment t, then this uniquely determines the future values of all the quantities – and uniquely determines the results of all future experiments and observations. In quantum physics, knowing the current state of the world allows us only to predict, for future experiments and observations, the *probabilities* of different results. At any given moment of time, these probabilities are uniquely determined based on the state of the quantum system – which is described by a *wave function* ψ. For the wave function, the change in time is still deterministic: if we know the state of the Universe at a given moment of time, then this information uniquely determines the state at all future moments of time [2, 5].

According to relativity theory, no processes can be faster than the speed of light c; see, e.g., [2, 5]. This means, in particular, that if we know the state of the world at moment t, then the values of all the physical quantities at a spatial point x at moment

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$t + \Delta t$ are only affected by the values of the physical quantities in a small vicinity (of radius $c \cdot \Delta t$) of the spatial point x. For "infinitesimal" (very small) Δt, the resulting change $q(x, t + \Delta t) - q(x, t)$ in the value of the quantity q is determined only by values $q(x + \Delta x, t)$ for infinitesimal Δx.

Infinitesimal changes is what calculus is about: e.g., the difference $q(x, t + \Delta t) - q(x, t)$ is usually interpreted as the product

$$\frac{\partial q}{\partial t} \cdot \Delta t.$$

So the above restriction is usually interpreted as meaning that the time derivative of q is determined by the value of this quantity and its spatial derivatives at the same spatial point. In general, a quantity itself may be a time derivative of other quantity – e.g., velocity is a time derivative of coordinates. Thus, in general, the changes of each field are described by *partial differential equations*.

The need to consider derivatives means that the fields are, in general, *smooth* – i.e., differentiable.

2 How Smooth – and Why This Question Is Important

From the mathematical viewpoint, an important question is how smooth the fields are; they could be:

- one time differentiable,
- two times differentiable,
- ...
- n times differentiable,
- ...
- infinitely many times differentiable, or
- *analytical*, i.e., locally representable as Taylor series.

From the viewpoint of practical physics, this mathematical question is not very relevant – physicists usually do not pay much attention to this and still get good predictions; see, e.g., [2, 5].

However, from the more fundamental viewpoint – e.g., from the viewpoint of analyzing whether the world is cognizable – this difference is very important. Indeed, it is theoretically true that if we know the values of all the quantities at all spatial locations at a given moment of time, then we can, in principle, predict the future state of the world. However, the same relativity theory – that encourages us to use differential equations – implies that we *cannot* know the current values of the fields at a given moment of time. For faraway spatial locations, we only know the values that happened long time ago. In practice, we can only know the values of the physical quantities in a reasonably small vicinity of our spatial location.

And here is where the difference between different degrees of smoothness becomes critical:

- in general, for a smooth functions, be it one time differentiable, two times differentiable, or even infinitely many times differentiable, knowing the value of this function in a bounded region does not help us determine its values outside this region: e.g., there exist many infinitely differentiable functions that are equal to 0 in a given bounded region and which are different from 0 outside this region;
- on the other hand, it is known that an analytical function is uniquely determined by its values in a bounded region.

So:

- if the fields are smooth but not analytical, our ability to predict the future is severely limited;
- on the other hand, if the fields are analytical, then, in principle, nothing prevents us from predicting all future events.

With this is mind, let us analyze whether the physical fields are only smooth or analytical. Since, as we have mentioned, this question is mostly studied in mathematics, we will use mathematical results to answer this question.

Comment. Of course, the fact that it is *theoretically possible* to predict the future state does not necessarily mean that it is, at this moment, *practically possible.* For many important problems, e.g., for predicting the path of a tornado in the next hours, current algorithms require computation time which is much longer than an hour – which makes these algorithms, at present, practically useless.

3 Smooth or Analytical: Towards Answering This Question

Of course, to seriously analyze this question, we need to analyze the corresponding partial differential equation, so the natural next question is: what are the actual partial differential equations?

There are many partial differential equations describing different physical phenomena. For example, to describe the dynamics of particles:

- first, there were Newton's equations,
- then it turned out that relativistic equations provide a more accurate picture, and
- then it turned out that we need to have quantum versions of these equations.

Based on such history, most physicists believe that the currently equations are only an approximation to reality, and that in the future, more accurate equations will appear – which, in their turn, will be replaced by more accurate ones, etc. As a result, we do not what are the actual equations. What can we do about it?

Actually, such a situation is rather typical in physics, when we do not know the actual function or the actual equation, but we need to make some conclusions. For example, this is a typical situation in statistical physics:

- it is theoretically possible, with a non-zero probability, that the thermal random motion of all the molecules in a human body will be oriented up, and the human will lift up by him/herself;

- however, the probability of this event is so low, that for almost all (in the common-sense meaning of this term) initial position of the molecules this is not possible.

So, while a mathematician may say that yes, this may happen once in many billions of billions of years, a physicists will definitely say that this lifting-up is *not* possible. According to physicists (and to common sense) a cold teapot places on a cold stove will *not* start boiling by itself – although theoretically, this is possible, with some very small probability.

On an even simpler example, if we flip a fair coin many times, it is theoretically possible that it will fall heads every time, but if this happens 1000 times in a row, it is reasonable to conclude that this coin is biased.

Physicists apply similar thinking outside statistical physics as well, e.g., to cosmology. How the Universe evolves depends on the initial conditions. For some initial conditions, we may have some weird scenarios, but for almost all initial conditions, we have a similar behavior – and this generic behavior is what physicists expect of the actual Universe; see, e.g., [4].

In general, if something is true for almost all initial conditions, physicists conclude that this property holds in real life as well.

4 Smooth or Analytical: Answer and Resulting Conclusion

In our case, what we do not know is not only the initial conditions, but also the sources of the fields, and the exact form of the corresponding differential equations. We can therefore apply the same reasoning and conclude that whatever is true for almost all equations and almost all functions describing sources and/or initial conditions – should be true in real life. And good news is there are mathematical results about it:

- a theorem proved by Hörmander (see, e.g., [1, 3]) states that for almost all partial differential equations and almost all smooth source functions, the equation has no solutions at all;
- on the other hand, the known 19 century Cauchy-Kowalewska result shows that when the source function is analytical, there is always a solution (at least locally).

Then, the above-described physical reasoning means that in real life, if we consider general smooth functions, the physical equations will have no solutions. Of course, to describe physical world, we need equations that have solutions.

So, the conclusion is that physical fields – that are sources for the corresponding equations – cannot be just smooth, they need to be analytical. And, as we have mentioned earlier, analytical means that we can, in principle, predict all future states based only on the available local information – in other words, it means that the world is, in principle, cognizable.

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