Towards Decision Making Under Interval Uncertainty

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Abstract In many real-life situations, we need to make a decision. In many cases, we know the optimal decision in situations when we know the exact value of the corresponding quantity x. However, often, we do not know the exact value of this quantity, we only know the bounds on the value x – i.e., we know the interval containing x. In this case, we need to select a decision corresponding to some value from this interval. The selected value will, in general, be different from the actual (unknown) value of this quantity. As a result, the quality of our decision will be lower than in the perfect case when we know the value x. Which value should we select in this case? In this paper, we provide a decision-theory-based recommendation for this selection.

1 Introduction

Situation. In many real-life situations, we need to make a decision.

The quality of the decision usually depends on the value of some quantity x. For example, in construction:

- the speed with which the cement hardens depends on the humidity, and
- thus, the proportions of the best cement mix depend on the humidity.

In practice, we often do not know the exact value of the corresponding quantity. For example, in the case of the pavement:

- while we can accurate measure the current humidity,
- what is really important is the humidity in the next few hours.

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For this future value, at best, we only know the bounds, i.e., we only know the interval $[\underline{x},\overline{x}]$ that contains the actual (unknown) value x. In other words, we have a situation of interval uncertainty; see, e.g., [2, 4, 7, 8],

Problem. To select a decision, we need to select some value x_0 from this interval and make the decision corresponding to this selection. Which value x_0 should we select?

What we do in this paper. In this paper, we describe a solution to this problem.

Comment. Results from this paper first appeared in [5].

2 Our Solution

General idea. In such situations of interval uncertainty, the ideal case is when the selected value x_0 is exactly equal to the actual value x. When these two values differ, i.e., when $x < x_0$ or $x > x_0$, the situation becomes worse.

In both cases when $x < x_0$ and when $x > x_0$, we have losses, but we often have two different reasons for a loss.

- If the humidity will be larger than expected, the hardening of the cement will take longer and we will lose time (and thus, money).
- In contrast, if the humidity is lower than expected, the cement will harden too fast, and the pavement will not be as stiff as it could be. So we will not get a premium for a good quality road (and we may even be required to repave some road segments).

In both cases, the larger the difference $|x - x_0|$, the larger the loss.

Possibility of linearization. The interval $[\underline{x},\overline{x}]$ is usually reasonable narrow, so the difference is small. In this case, the dependence of the loss on the difference can be well approximated by a linear expression; so:

- when $x < x_0$, the loss is $\alpha_- \cdot (x_0 - x)$ for some α_-, and
- when $x > x_0$, the loss is $\alpha_+ \cdot (x - x_0)$ for some α_+.

Resulting formula for the worst-case loss.

- When $x < x_0$, the worst-case loss is when x is the smallest: $\alpha_- \cdot (x_0 - \underline{x})$.
- When $x > x_0$, the worst-case loss is when x is the largest: $\alpha_+ \cdot (\overline{x} - x_0)$.

In general, the worst-case loss is the largest of these two:

$$w(x_0) = \max(\alpha_- \cdot (x_0 - \underline{x}), \alpha_+ \cdot (\overline{x} - x_0)).$$

Range (interval) of possible values of the loss. The best-case loss is 0 – when we guessed the value x correctly. In this case, all we know is that the loss is somewhere between 0 and $w(x_0)$. So, the gain is somewhere between $\underline{g} = -w(x_0)$ and $\overline{g} = 0$.

Let us use Hurwicz criterion. In situations where we only know the interval of possible values of the gain, decision theory recommends to use Hurwicz optimism-pessimism criterion to make a decision [1, 3, 6], i.e.:

- to select some value $\alpha > 0$ and
- then to select an alternative for which the value $g \stackrel{\text{def}}{=} \alpha \cdot \overline{g} + (1-\alpha) \cdot \underline{g}$ is the largest possible.

In our case, $g = -(1-\alpha) \cdot w(x_0)$, so maximizing g simply means selecting the value x_0 for which $w(x_0)$ is the smallest.

Let us solve the resulting optimization problem. Here, the value $\alpha_- \cdot (x_0 - \underline{x})$ increases with x_0, while the value $\alpha_+ \cdot (\overline{x} - x_0)$ decreases with x_0. Thus, the function $w(x_0)$ – which is the minimum of these two expressions:

- decreases until the point $\widetilde{x}$ at which these two expressions coincide, and
- then increases.

So, the minimum of the worst-case loss $w(x_0)$ is attained at the point $\widetilde{x}$ for which $\alpha_- \cdot (\widetilde{x} - \underline{x}) = \alpha_+ \cdot (\overline{x} - \widetilde{x})$, i.e., for $\widetilde{x} = \widetilde{\alpha} \cdot \overline{x} + (1 - \widetilde{\alpha}) \cdot \underline{x}$. Here, we denoted

$$\widetilde{\alpha} \stackrel{\text{def}}{=} \frac{\alpha_+}{\alpha_+ + \alpha_-}.$$

Comment. Interestingly, we get the same expression as with the Hurwicz criterion!

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References

1. L. Hurwicz, *Optimality Criteria for Decision Making Under Ignorance*, Cowles Commission Discussion Paper, Statistics, No. 370, 1951.

2. L. Jaulin, M. Kiefer, O. Didrit, and E. Walter, *Applied Interval Analysis, with Examples in Parameter and State Estimation, Robust Control, and Robotics*, Springer, London, 2001.
3. V. Kreinovich, "Decision making under interval uncertainty (and beyond)", In: P. Guo and W. Pedrycz (eds.), *Human-Centric Decision-Making Models for Social Sciences*, Springer Verlag, 2014, pp. 163–193.
4. B. J. Kubica, *Interval Methods for Solving Nonlinear Constraint Satisfaction, Optimization, and Similar Problems: from Inequalities Systems to Game Solutions*, Springer, Cham, Switzerland, 2019.
5. J. A. Lopez and V. Kreinovich, "Towards Decision Making Under Interval Uncertainty", *Abstracts of the NMSU/UTEP Workshop on Mathematics, Computer Science, and Computational Science*, Las Cruces, New Mexico, April 1, 2023.
6. R. D. Luce and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.
7. G. Mayer, *Interval Analysis and Automatic Result Verification*, de Gruyter, Berlin, 2017.
8. R. E. Moore, R. B. Kearfott, and M. J. Cloud, *Introduction to Interval Analysis*, SIAM, Philadelphia, 2009.