What Do Goedel's Theorem and Arrow's Theorem Have in Common: A Possible Answer to Arrow's Question

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Abstract Kenneth Arrow, the renowned author of the Impossibility Theorem that explains the difficulty of group decision making, noticed that there is some commonsense similarity between his result and Goedel's theorem about incompleteness of axiomatic systems. Arrow asked if it is possible to describe this similarity in more precise terms. In this paper, we make the first step towards this description. We show that in both cases, the impossibility result disappears if we take into account probabilities. Namely, we take into account that we can consider probabilistic situations, that we can make probabilistic conclusions, and that we can make probabilistic decisions (when we select different alternatives with different probabilities).

1 Formulation of the Problem

Need to consider the axiomatic approach. Both Goedel's and Arrow's results are based on axioms. So, to analyze possible similarities between these two results, let us recall where the axiomatic approach came from.

Already ancient people had to deal the measurements of lengths, angles, areas, and volumes: this was important in constructing building, it was important for deciding how to re-mark borders between farms after a flood, etc.

The experience of such geometric measurements led to the discovery of many interesting relations between all these quantities. For example, people empirically discovered what we now call Pythagoras theorem – that in a right triangle, the square of the hypothenuse is equal to the sum of the square of the sides.

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At first, this was just a collection of interesting and useful relations. After a while, people noticed that some of these relations can be logically deduced from others. Eventually, it turned out that it is enough to state a small number of these relations – these selected relations became known as *axioms* – so that every other geometric relation, every other geometric fact, every other statement about geometric objects can be deduced from these axioms.

In general, it was believed that for each geometric statement, we can either deduce this statement or its negation from the axioms of geometry.

Can axiomatic method be applied to arithmetic? The success of axiomatic method in geometry led to a natural idea of using this method in other disciplines as well. Since geometry is part of mathematics, a natural idea is to apply axiomatic method to other parts of mathematics, starting with statements about natural numbers.

The corresponding axioms have indeed been formulated in the 19th century. Most mathematicians believed that – similarly to geometry – for each statement about natural numbers, we can either deduce this statement or its negation from these axioms. OK, some mathematicians were not 100% sure that the available axioms would be sufficient for this purpose, but they were sure that in this case, we can achieve a similar result if we add a few additional axioms.

Goedel's impossibility result. Surprisingly, it turned out that the desired complete description of natural numbers is not possible: no matter what axioms we select:

- either there is a statement for which neither this statement not its negation can be derived from the axioms,
- or the set of axioms is inconsistent – i.e., for each statement, we can deduce *both* this statement and its negation from these axioms.

In other words, it is not possible to formulate the set of axioms about natural numbers that would satisfy the natural condition of completeness. This result was proven by Kurt Goedel [5].

In particular, Goedel proved that the incompleteness can already be shown for statements of the type $\forall n\, P(n)$ for algorithmically decidable formulas $P(n)$.

Arrow's impossibility result. In addition to mathematics, there is another area of research where we have reasonable requirements: namely, human behavior. Already Baruch Spinoza tried to describe human behavior in axiomatic terms; see, e.g., [14].

In the 20th century, researchers continued such attempts. In particular, attempts were made to come up with a scheme of decision making that would satisfy natural fairness restrictions. Similar to natural numbers, at first, researchers hoped that such a scheme would be possible. However, in 1951, the future Nobelist Kenneth Arrow proved his famous Impossibility Theory: that it is not possible to have an algorithm that, given preferences of several participants would come up with a group decision that would satisfy natural fairness requirements; see, e.g., [1]. For this result, Arrow was awarded the Nobel prize.

What do Goedel's and Arrow's impossibility theorems have in common: Arrow's question. From the commonsense viewpoint, these two results are somewhat similar: they are both about impossibility of satisfying seemingly reasonable conditions. However, from the mathematical viewpoint, these two results are very different: the formulations are different, the proofs are different, etc.

The fact that from the commonsense viewpoint, these results are similar made Arrow conjecture that there must be some mathematical similarity between these two results as well; see, e.g., [2].

What we do in this paper. In this paper, we show that, yes, there is some mathematical similarity between these two results.

Our result just scratches the surface, but we hope that it will lead to discovering deeper and more meaningful similarities.

2 Our Main Idea and the Resulting Similarity

Implicit assumption underlying Arrow's result. Arrow's result implicitly assumed that all our decisions are deterministic. But is this assumption always true?

But is this assumption true? Not really. The fact that this assumption is somewhat naive can be best illustrated by the known argument about a donkey – first described by a philosopher Buridan. According to this argument, a donkey placed between two identical heaps of hay will not be able to select one of them and will, thus, die of hunger.

Of course, the real donkey will not die, it will select one of the heaps at random and start eating the juicy hay. Similarly, a human when facing a fork in a road – without any information about possible paths, will randomly select one of the two directions. When two friends meet for dinner and each prefers his own favorite restaurant, they will probably flip a coin to decide where to eat.

In other words, people not only make deterministic decisions, they sometimes make probabilistic decisions, i.e., they select different actions with different probabilities.

Let us take probabilities into account when describing preferences and decisions. Since probabilistic decisions are possible, we need to take them into account – both when we describe preferences and when we select a decision. We will show that taking into account – that leads to so-called *decision theory* [3, 4, 6, 7, 10, 11, 12] – helps to resolve the paradox of Arrow's theorem, namely, leads to a reasonable way to select a joint decision. From this viewpoint:

- Arrow's theorem means that if we only know preferences between *deterministic* alternatives, then we cannot consistently select a *deterministic* fair joint action;
- however, we will show that if we take into account preferences between *probabilistic* options and allow *probabilistic* joint decisions, then a fair solution is possible.

Let us take probabilities into account when describing preferences. First, we need to show how we can describe the corresponding preferences. Let us select two alternatives:

- a very good alternative A_+ which is better than anything that we will actually encounter and
- a very bad alternative A_- which is worse than anything that we will actually encounter.

Then, for each real number p from the interval $[0, 1]$, we can consider a probabilistic alternative ("lottery") $L(p)$ in which we get A_+ with probability p and A_- with the remaining probability $1 - p$. For each actual alternative A, we can ask the user to compare this alternative A with lotteries $L(p)$ corresponding to different probabilities p.

Here:

- For small p, the lottery is close to A_- and is, thus, worse than A; we will denote it by $L(p) < A$.
- For probabilities close to 1, the lottery $L(p)$ is close to A_+ and is, thus, better than A: $A < L(p)$.

Also:

- If $L(p) < A$ and $p' < p$, then clearly $L(p') < A$.
- Similarly, if $A < L(p)$ and $p < p'$, then $A < L(p')$.

One can show that in this case, there exists a threshold value u such that:

- for all $p < u$, we have $L(p) < A$, while
- for all $p > u$, we have $A < L(p)$:

$$\inf\{p : L(p) < A\} = \sup\{p : A < L(p)\}.$$

In this case, for arbitrarily small $\varepsilon > 0$, we have $L(u - \varepsilon) < A < L(u + \varepsilon)$. Since in practice, we can only set up probability with some accuracy, this means that, in effect, the alternative A is equivalent to $L(u)$. This threshold value u is called the *utility* of the alternative A. Utility of A is denoted by $u(A)$.

Utility is not uniquely determined. The numerical value of the utility depends on the selection of the two alternatives A_- and A_+. One can show that if we select two different alternatives A'_- and A'_+, then the new utility value is related to the original utility value by a linear transformation $u'(A) = a \cdot u(A) + b$ for some $a > 0$ and b.

Utility of a probabilistic alternative. One can also show that the utility of a situation in which we get alternatives A_i with probabilities p_i is equal to

$$p_1 \cdot u(A_1) + p_2 \cdot u(A_2) + \ldots$$

How to make a group decision. Suppose now that n participants need to make a joint decision. There is a status quo state A_0 – the state that will occur if we do not make any decision. Since utility of each participant is defined modulo a linear transformation, we can always apply a shift $u(A) \mapsto u(A) - u(A_0)$ and thus, get the utility of the status quo state to be equal to 0. Therefore, preferences of each participant i can be described by the utility $u_i(A)$ for which $u_i(A_0) = 0$.

These utilities are defined modulo linear transformations $u_i(A) \mapsto a_i \cdot u_i(A)$ for some a_i. It therefore makes sense to require that the group decision making should not change if we thus re-scale each utility value. It turns out that the only decision making that does not change under this re-scaling means selecting an alternative A with the largest product of the utilities $u_1(A) \cdot \ldots \cdot u_n(A)$. This result was first shown by a Nobelist John Nash and is thus known as Nash's bargaining solution; see, e.g., [7, 8, 9]. If we allow probabilistic combinations of original decisions, then such a solution is, in some reasonable sense, unique.

Adding probabilities to Arrow's setting: conclusion. One can show that Nash's bargaining solution satisfies all fairness requirements. So, in this case, we indeed have a solution to the group decision problem – exactly what, according to Arrow's result, is not possible if we do not take probabilities into account.

What about Goedel's theorem? Goedel's theorem states that, no matter what finite list of axioms we choose, for some true statements of the type $\forall n\, P(n)$, we will never deduce the truth of this statement from these axioms. Let us look at this situation from the commonsense viewpoint.

The fact that the statement $\forall n\, P(n)$ is true means that for all natural numbers n, we can check that the property $P(n)$ is true. We can check it for $n = 0$, we can check it for $n = 1$, etc., and this will be always true. And this is exactly how we reason about the properties of the real world:

- someone proposes a new physical law,
- we test it many times,
- every time, this law is confirmed,
- so we start believing that this law is true.

The more experiments we perform, the higher our subjective probability that this law is true. If a law is confirmed in n experiments, statistics estimates it as $1 - 1/n$; see, e.g., [13].

So, yes, we can never become 100% sure (based on the axioms) that the statement $\forall n\, P(n)$ is true, but are we ever 100% sure? Even for long proofs, there is always a probability that we missed a mistake – and sometimes such mistakes are found. We usually trust results of computer computations, but sometimes, some of the computer cells goes wrong, and this makes the results wrong – such things also happened.

So, in this case too, allowing probabilistic outcomes also allows us to make practically definite conclusions – contrary to what happens when we are not taking probabilities into account, in which case Goedel's theorem shows that conclusions are, in general, not possible.

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