How People Make Decisions Based on Prior Experience: Formulas of Instance-Based Learning Theory (IBLT) Follow from Scale Invariance

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Abstract To better understand human behavior, we need to understand how people make decisions, how people select one of possible actions. This selection is usually based on predicting consequences of different actions, and these predictions are, in their turn, based on the past experience. For example, consequences that occur more frequently in the past are viewed as more probable. However, this is not just about frequency: recent observations are usually given more weight that past ones. Researchers have discovered semi-empirical formulas that describe our predictions reasonably well; these formulas form the basis of the Instance-Based Learning Theory (IBLT). In this paper, we show that these semi-empirical formulas can be derived from the natural idea of scale invariance.

1 Formulation of the Problem

How do people make decisions? To properly make a decision, i.e., to select one of the possible actions, we need to predict the consequences of each of these actions. To predict the consequences of each action, we take into account past experience, in which we know the consequences of similar actions. Often, at different occasions, the same action led to different consequences. So, we cannot predict what exactly will be the consequence of each action. At best, for each action, we can try to predict the probability of different consequences.

In this prediction, we take into account the frequency with which each consequence occurred in the past. We also take into account that situations change, so more recent observations should be given more weight than the ones that happened long ago. To better understand human behavior, we need to know how people take all this into account.

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Semi-empirical formulas. By performing experiments and by analyzing the resulting data, researchers found some semi-empirical formulas that provide a very good description of the actual human behavior [4, 5] (see also [2]). These formulas form the basis of the Instance-Based Learning Theory (IBLT).

In the first approximation, when we only consider completely different consequences, these formulas have the following form. For each possible action, to estimate the probability p_i of each consequence i, we first estimate the activation A_i of this consequence as

$$A_i = \ln\left(\sum_j (t - t_{i,j})^{-d}\right), \tag{1}$$

where:

- t is the current moment of time (i.e., the moment of time at which we make a decision),
- the values $t_{i,1}$, $t_{i,2}$, etc. are past moments of time at which the same action led to consequence i, and
- $d > 0$ is a constant – depending on the decision maker.

Based on these activation values, we estimate the probability p_i as

$$p_i = \frac{\exp(c \cdot A_i)}{\sum_k \exp(c \cdot A_k)}, \tag{2}$$

where c is another constant depending on the decision maker, and the summation in the denominator is over all possible consequences k.

Challenge. How can we explain why these complex formulas properly describe human behavior?

What we do in this paper. In this paper, we show that these formulas can be actually derived from the natural idea of scale invariance.

2 Our Explanation

Analysis of the problem. If time was not the issue, then the natural way to compare different consequences i would be by comparing the number of times n_i that the i-th consequence occurred in the past:

$$n_i = \sum_j 1. \tag{3}$$

In this formula, for each past observation of the i-th consequence, we simply add 1. In other words, all observations are assigned the same weight.

As we have mentioned, it makes sense to provide larger weight to more recent observations and smaller weight to less recent ones. In other words, instead of adding

1s, we should add some weights depending on the time $\Delta t = t - t_{i,j}$ that elapsed since the observation. Let us denote the dependence on the weight on time by $f(\Delta t)$. This function should be decreasing with Δt: the more time elapsed, the smaller the weight. In these terms, the simplified formula (3) should be replaced by the following more adequate formula

$$n_i = \sum_j f(t - t_{i,j}). \tag{4}$$

Based on the corresponding values n_i – describing the time-adjusted number of observations – we need to predict the corresponding probabilities p_i. The more frequent the consequence, the higher should be its probability. At first glance, it may seem that we can simply take $p_i = g(n_i)$ for some increasing function $g(z)$. However, this will not work, since the sum of the probabilities of different consequences should be equal to 1. To make sure that this sum is indeed 1, we need to "normalize" the values $g(n_i)$, i.e., to divide each of them by their sum:

$$p_i = \frac{g(n_i)}{\sum\limits_k g(n_k)}. \tag{5}$$

Remaining question. The above formulas (4) and (5) leave us with a natural question: which functions $f(t)$ and $g(z)$ better describe human behavior?

Our main idea: scale-invariance. To answer this question, let us take into account that the numerical value of time duration – as well as the numerical values of many other physical quantities – depends on the choice of the measuring unit. If we replace the original unit for measuring time by a new unit which is $\lambda > 0$ times smaller, then all numerical values of time intervals get multiplied by λ: $t \mapsto \lambda \cdot t$. For example, if we replace minutes by seconds, then all numerical values are multiplied by 60, so that, e.g., 2 minutes becomes 120 seconds.

In many physical (and other) situations, there is no physically preferred unit for measuring time intervals x. This means that the formulas should remain the same if we "re-scale" t by choosing a different measuring unit, i.e., by replacing all numerical values t with $t' = \lambda \cdot t$. This "remains the same" is called scale invariance.

Now, we are ready to formulate our main result.

Proposition. *Let $f(x)$ and $g(x)$ by continuous monotonic functions. Then, the following conditions are equivalent to each other:*

- *for each $\lambda > 0$, the values of p_i as described by the formulas* (4) *and* (5) *will remain the same if we replace t and $t_{i,j}$ with $t' = \lambda \cdot t$ and $t'_{i,j} = \lambda \cdot t_{i,j}$;*
- *the dependence* (4)-(5) *is described by the formulas* (1)-(2).

Proof.

1°. Let us first show that if we re-scale all the time values in the formulas (1)-(2), the probabilities remain the same.

Indeed, in this case,

$$\sum_j (t' - t'_{i,j})^{-d} = \lambda^{-d} \cdot \sum_j (t - t_{i,j})^{-d}$$

and thus, the new value A'_i of the activity is:

$$A'_i = \ln\left(\sum_j (t' - t'_{i,j})^{-d}\right) = \ln\left(\sum_j (t - t_{i,j})^{-d}\right) + \ln(\lambda^{-d}) = A_i + \ln(\lambda^{-d}).$$

Thus, we have

$$c \cdot A'_i = c \cdot A_i + c \cdot \ln(\lambda^{-d}),$$

and $\exp(c \cdot A'_i) = C \cdot \exp(c \cdot A_i)$, where we denoted $C \stackrel{\text{def}}{=} \exp(c \cdot \ln(\lambda^{-d}))$. Hence, the new expression for probability takes the form

$$p'_i = \frac{\exp(c \cdot A'_i)}{\sum\limits_k \exp(c \cdot A'_k)} = \frac{C \cdot \exp(c \cdot A_i)}{C \cdot \sum\limits_k \exp(c \cdot A_k)}.$$

If we divide both the numerator and the denominator of the right-hand side by the same constant C, we conclude that $p'_i = p_i$, i.e., that the probabilities indeed do not change.

2°. So, to complete our proof, it is sufficient to prove that if the expressions (4)-(5) leads to scale-invariant probabilities, then the dependence (4)-(5) is described by the formulas (1)-(2).

2.1°. Let us first find what we can deduce from scale-invariance about the function $g(x)$.

To do that, let us consider the case when we have two consequences, one of which was observed only once, and the other one was observed m times for some $m > 1$. Let us also assume that all the observations occurred at the same time Δt moments in the past, so that $t - t_{i,j} = \Delta t$. In this case, $n_1 = f(\Delta t)$, $n_2 = m \cdot n_1$, and the formula for the probability p_1 takes the form

$$p_1 = \frac{g(f(\Delta t)}{g(m \cdot f(\Delta t)) + g(f(\Delta t))}. \quad (6)$$

By re-scaling time, we can replace Δt with any other value, and this should not change the probabilities. Thus, the formula (6) should retain the same value for all possible values of $z = f(\Delta t)$. In other words, the ratio

$$\frac{g(z)}{g(m \cdot z) + g(z)}$$

should not depend on z, it should only depend on m. Hence, its inverse

$$\frac{g(m \cdot z) + g(z)}{g(z)} = \frac{g(m \cdot z)}{g(z)} + 1$$

should also depend only on m, and therefore, that we should have

$$\frac{g(m \cdot z)}{g(z)} = a(m)$$

for some function $a(m)$. So, we should have $g(m \cdot z) = a(m) \cdot g(z)$ for all z and m.

In particular, for $z' = z/m'$ for which $m' \cdot z' = z$, we should have $g(m' \cdot z') = a(m') \cdot g(z')$, i.e., $g(z) = a(m') \cdot g(z/n')$, hence $g(z/n') = (1/a(m')) \cdot g(z)$. So, for each rational number $r = m/m'$, we should have

$$g(r \cdot z) = g(m \cdot (z/m')) = a(m) \cdot g(z/m') = a(m) \cdot (1/a(m')) \cdot g(z).$$

In other words, for every z and for every rational number r, we should have

$$g(r \cdot z) = a(r) \cdot g(z), \tag{7}$$

where we denoted $a(r) \stackrel{\text{def}}{=} a(m) \cdot (1/a(m'))$.

By continuity, we can conclude that the formula (7) should hold for all real values r, not necessarily for rational values. It is known that every continuous solution to the functional equation (7) is the power law, i.e., it has the form $y = A \cdot x^a$ for some constants A and a; see, e.g., [1]. Thus, we conclude that

$$g(z) = A \cdot z^a. \tag{8}$$

2.2°. We can simplify the resulting expression (8) even more.

Indeed, substituting the expression (8) into the formula (5), we conclude that

$$p_i = \frac{A \cdot n_i^a}{A \cdot \sum_k n_k^a}.$$

We can simplify this expression if we divide both the numerator and the denominator by the same constant A. Then, we get the following simplified formula

$$p_i = \frac{n_i^a}{\sum_k n_k^a} \tag{9}$$

that corresponds to the function $g(z) = z^a$. So, without loss of generality, we can conclude that $g(z) = z^a$.

2.3°. Let us now find out what we can deduce from scale-invariance about the function $f(t)$.

For this purpose, let us consider two consequences each of which was observed exactly once, one of which was observed 1 time unit ago and the other one was observed t_0 time units ago. Then, according the formulas (4) and (9), the predicted probability p_1 should be equal to

$$p_1 = \frac{(f(1))^a}{(f(1))^a + (f(t_0))^a}.$$

By scale-invariance, this probability should not change if we multiply both time intervals by λ, so that $1 \mapsto \lambda$ and $t_0 \mapsto \lambda \cdot t_0$:

$$\frac{(f(1))^a}{(f(1))^a + (f(t_0))^a} = \frac{(f(\lambda))^a}{(f(\lambda))^a + (f(\lambda \cdot t_0))^a}.$$

The equality remains valid if we take the inverses of both sides:

$$\frac{(f(1))^a + (f(t_0))^a}{(f(1))^a} = \frac{(f(\lambda))^a + (f(\lambda \cdot t_0))^a}{(f(\lambda))^a},$$

subtract 1 from both sides, resulting in:

$$\frac{(f(t_0))^a}{(f(1))^a} = \frac{(f(\lambda \cdot t_0))^a}{(f(\lambda))^a},$$

and raise both sides to the power $1/a$:

$$\frac{f(t_0)}{f(1)} = \frac{f(\lambda \cdot t_0)}{f(\lambda)}.$$

The left-hand side of this equality does not depend on λ, it depends only on t_0. Thus, the right-hand side should also depend only on t_0, i.e., we should have

$$\frac{f(\lambda \cdot t_0)}{f(\lambda)} = F(t_0),$$

for some function $F(t_0)$. Multiplying both sides by $f(\lambda)$, we conclude that

$$f(\lambda \cdot t_0) = F(t_0) \cdot f(\lambda)$$

for all $t_0 > 0$ and $\lambda > 0$.

We have already mentioned that every continuous solution to this functional equation has the form

$$f(t) = B \cdot t^b \tag{10}$$

for some constants B and b. Since the function $f(t)$ is decreasing, we have $b < 0$.

2.4°. We can simplify the expression (10) even more.

Indeed, substituting the expression (10) into the formula (4), we get

$$n_i = B \cdot \sum_j (t - t_{i,j})^b,$$

i.e., $n_i = B \cdot a_i$, where we denoted

$$a_i \stackrel{\text{def}}{=} \sum_k (t - t_{i,j})^b. \tag{11}$$

Substituting the formula $n_i = B \cdot a_i$ into the formula (9), we get

$$p_i = \frac{B^a \cdot a_i^a}{B^a \cdot \sum\limits_k a_k^a}.$$

We can simplify this expression if we divide both the numerator and the denominator by the same constant B^a. Then, we get the following simplified formula

$$p_i = \frac{a_i^a}{\sum\limits_k a_k^a}$$

that corresponds to using a_i instead of n_i, i.e., in effect, to using the function $f(t) = t^b$. So, without loss of generality, we can conclude that $f(t) = t^b$. For this function $f(t)$, we have

$$p_i = \frac{n_i^a}{\sum\limits_k n_k^a}. \tag{11}$$

2.5°. Let us show that for $f(t) = t^b$ for $b < 0$ and $g(z) = z^a$, we indeed get the expression (1)-(2).

Indeed, for $f(t) = t^b$, the formula (4) takes the form

$$n_i = \sum_j (t - t_{i,j})^b,$$

i.e., the form

$$n_i = \sum_j (t - t_{i,j})^{-d},$$

where we denoted $d \stackrel{\text{def}}{=} -b$. Thus, the expression (1) takes the form $A_i = \ln(n_i)$. So, the expression $\exp(c \cdot A_i)$ in the empirical formula (2) takes the form

$$\exp(c \cdot A_i) = \exp(c \cdot \ln(n_i)) = (\exp(\ln(n_i))^c = n_i^c.$$

Thus, the formula (2) takes the form

$$p_i = \frac{n_i^c}{\sum\limits_k n_k^c}.$$

One can see that this is exactly our formula (11), the only difference is that the parameters that is denoted by c in the formula (2) is denoted a in the formula (11).

Thus, we have indeed explained the empirical formulas (1) and (2). The proposition is proven.

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