Natural Color Interpretation of Interval-Valued Fuzzy Degrees

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Abstract

Intuitively, interval-values fuzzy degrees are more adequate for representing expert uncertainty than the traditional $[0,1]$-based ones. Indeed, the very need for fuzzy degrees comes from the fact that experts often cannot describe their opinion not in terms of precise numbers, but by using imprecise ("fuzzy") words from natural language like "small". In such situations, it is strange to expect the same expert to be able to provide an exact number describing his/her degree of certainty; it is more natural to ask this expert to mark the whole interval (or even, more generally, a fuzzy set of possible degrees). In spite of this intuitive adequacy, and in spite of several successful applications of interval-valued degrees, most applications of fuzzy techniques are still based on the traditional $[0,1]$-based degrees. According to researcher who studied this puzzling phenomenon, the problem is that while people are accustomed to marking their opinion on a numerical scale, most people do not have any experience of using interval. To ease people's use of interval-valued degrees, we propose to take into account that the set of all interval-valued degrees is, in some reasonable sense, equivalent to the set of colors – thus, we can represent degrees as appropriate colors. This idea can be naturally extended to Z-numbers – and it also provides an additional argument why interval-valued degrees are more adequate, at least more adequate in the analysis of complex phenomena.

Keywords: Interval-valued fuzzy, Optical computing, Z-numbers

1 Formulation of the Problem

Fuzzy knowledge is important. A significant part of human knowledge – including expert knowledge in many application areas – is described in terms of imprecise ("fuzzy") words from natural language. For example, medical instructions are filled with words like "high fever", "high blood pressure", "low red blood cell count".

- There are often precise thresholds – e.g., the precise 38° C threshold for fever.
- However, it is very clear that there is no critical difference between temperatures 37.9 and 38.0.

As a result, in such borderline cases, medical doctors do not blindly follow the formalized rules: instead, they take into account the whole state of the patient when prescribing treatment.

Traditional $[0,1]$-based approach to describing fuzzy knowledge. To describe imprecise (fuzzy) knowledge in computer-understandable numerical terms, Lotfi Zadeh suggested to describe, for each corresponding statement, the degree to which this statement is true. For example:

- 38.5 is clearly a fever (with degree 1),
- 36.6 is clearly not a fever (i.e., a fever with degree 0), while
- 37.8 is a fever with some degree intermediate between 0 and 1.

This idea started the currently well developed area of fuzzy techniques and their applications (see, e.g., [2, 5, 7, 8, 9, 15]), with many successful applications.

Need to go beyond the traditional $[0,1]$-based logic. In the traditional fuzzy technique, to each imprecise natural-language property P (like "high") and to each possible value x of the corresponding quantity (e.g., "temperature"), we ask the expert to assign a degree $P(x)$ to which the value x has the property P.

This idea works successfully in many applications, but it does not fully capture our intuitive idea of fuzziness and imprecision. Indeed:

- we start with the correct idea that users *cannot* come up with an *exact* threshold separating high and from high temperature is high and what is not
- and then we require the same users to come up with the *exact* number describing their degree of certainty.

Intuitively:

- we can usually meaningfully distinguish between degrees 0.6 and 0.8,
- but hardly anyone can meaningfully distinguish between degrees 0.8 and 0.81.

It seems more natural to take into account that we can only assign degrees with uncertainty – e.g., describe the range (interval) of possible degrees.

Allowing us to consider intervals of degrees helps resolve another problems associated of the traditional $[0,1]$-based description: that in this description, there is no easy way to distinguish between:

- the case when we have no information about the situation and
- the case when we have many arguments in favor of the statements and equally many arguments against.

In the $[0,1]$-based approach, both cases are described by the same degree 0.5 – the degree which is exactly in the middle between 0 and 1. In contrast, in the interval-valued case:

- we can keep the degree 0.5 (i.e., the interval $[0.5,0.5]$) for the case when we have many arguments for and equally many arguments against,
- while we can naturally assign the interval $[0,1]$ – indicating that anything is possible – to the case when we have no information at all.

If interval-valued fuzzy techniques are so promising, why not everyone is using them? On a theoretical level, it seems that interval-valued techniques provide a more adequate description of uncertainty. However, in practice, most applications of fuzzy techniques still use the traditional $[0,1]$-based values. Why?

A possible answer comes from the paper [4] which explains that:

- while people are accustomed to marking a value on a scale – we do it for surveys, students do it for faculty evaluations, we do it to indicate our opinion about the quality of a restaurant or a movie,
- most people are not accustomed to marking intervals.

To overcome this issue, the authors of [4] proposed an innovative techniques of asking users to make an ellipsoid containing the desired interval (see also [17]). This helps, but still interval-valued techniques are far from being universally accepted. This leads to the following problem.

Problem. We need to make interval-valued techniques more natural, more convenient for people to use.

2 Our Idea: Color Representation

Geometric analysis of the problem. In the traditional approach to fuzzy uncertainty, possible degree if certainty are numbers from the interval $[0,1]$. In geometric terms, this interval can be described as follows:

- we start with two degrees 0 and 1, and
- we consider all possible convex combinations of these two degrees, i.e., all possible combinations of the type $\alpha_1 \cdot 1 + \alpha_0 \cdot 0$ for which $\alpha_i \geq 0$ and

$$\alpha_0 + \alpha_1 = 1.$$

Indeed, any value a from the interval $[0,1]$ can be represented in this form, with $\alpha_0 = a$ and $\alpha_1 = 1 - a$.

Interval-valued degrees can be described in a similar way:

- we have three degrees: 0 (i.e., $[0,0]$), 1 (i.e., $[1,1]$), and $[0,1]$, and
- we consider all possible convex combinations of these three degrees, i.e., all possible combinations of the type

$$\alpha_1 \cdot [1,1] + \alpha_0 \cdot [0,0] + \alpha_{01} \cdot [0,1] \qquad (1)$$

for which $\alpha_i \geq 0$ and $\alpha_0 + \alpha_1 + \alpha_{01} = 1$.

In this formula, addition and multiplication are understood component-wise, i.e.,

- $\alpha \cdot [a, b]$ is understood as $[\alpha \cdot a, \alpha \cdot b]$, and
- $[a_1, b_1] + [a_2, b_2]$ is understood as
$$[a_1 + a_2, b_1 + b_2].$$

In this interpretation, the formula (1) leads to the interval

$$[\alpha_1 \cdot 1 + \alpha_0 \cdot 0 + \alpha_{01} \cdot 0, \alpha_1 \cdot 1 + \alpha_0 \cdot 0 + \alpha_{01} \cdot 1] =$$
$$[\alpha_1, \alpha_1 + \alpha_{01}].$$

Thus, to represent any interval $[a, b] \subseteq [0, 1]$ in this form, it is sufficient to select the values α_i for which $\alpha_1 = a$, $\alpha_1 + \alpha_{01} = b$, and $\sigma_0 + \alpha_1 + \alpha_{01} = 1$.

Thus, we can take $\alpha_1 = a$, $\alpha_{01} = b - a$ and, correspondingly, $\alpha_0 = 1 - (\alpha_1 + \alpha_{01}) = 1 - b$.

In view of this geometric representation, how can we make it natural? In line with this geometric representation of interval-valued fuzzy techniques, how can make this natural? To look for such a natural representation, let us look for some natural phenomenon in which we have 3 basic states and every other state can be represented as a convex combination of these basic states.

Of course, there is such a phenomenon – it is the phenomenon of color:

- there are three basic colors, e.g., red, green, and blue, and
- every color can be represented as a convex combination of these colors.

This is not just a theoretical idea:

- this is how our eyes perceive color, via cells attuned to these three basic colors,
- this is how computer screens represent color – by three dense grids of light sources corresponding to three basic colors, etc.

So, we arrive at the following idea.

Main idea. Let us represent each fuzzy interval $[a, b]$ by an appropriate color, namely, by a convex combination of three basic colors with coefficients a, $b - a$ and $1 - b$.

Details. Which of the three basic colors should correspond to the three basic elements 0, 1, and $[0, 1]$? Which basic colors should corresponding to 0, which to 1, and which to $[0, 1]$?

From the commonsense viewpoint, the interval $[0, 1]$ is kind of between 0 ("absolutely false") and 1 ("absolutely true"). Thus, as a color corresponding to $[0, 1]$, it is reasonable to select a color which is, in some reasonable sense, intermediate between the two others. For colors, there is a natural linear order, corresponding to comparison of the frequency of the corresponding light. In this order:

- red color has the smallest frequency,
- blue has the largest frequency, and
- green is in between.

Thus, it is reasonable to associate green with the interval $[0, 1]$.

How do we interpret the remaining two colors: red and blue? One of them should correspond to 0, another to one. With street lights, we are so accustomed to associating red with prohibition and negation that it is reasonable to associate red with 0 (false) and blue with 1 (true).

Comment. Another reason for such an association is that "true" usually carries more information than "false" – e.g., when we try to explain a natural phenomenon, most explanation attempts turns our top be false, and only one is true. From this viewpoint, it is reasonable to associate 1 (true) with the color that carries more information. From this viewpoint, since each cycle can convey a certain amount of information, the information-carrying capacity of each color is proportional to its frequency. Since the blue color corresponds to higher frequency, it is more informative and thus, should be associated with 1.

Resulting color representation of interval-valued fuzzy degrees. So, we arrive at the following representation of an interval $[a, b] \subseteq [0, 1]$: it is a convex combination, in which we have:

- blue with coefficient b,
- green with coefficient $b - a$, and
- red with coefficient a.

Comment. Our preliminary tests show that this scheme indeed provides users with a reasonable representation of uncertainty.

Color representation can be used not only to represent fuzzy degrees, but also to process them. Up to now, we explained that colors provide a natural *representation* of interval-valued fuzzy degrees. Interestingly, the color representation can be also used for *processing* fuzzy degrees: namely, we can process the corresponding colored beams of light; see, e.g., [10, 11, 12, 13, 14].

The main advantage of such optical data processing – and optical data processing in general – is that it can be naturally parallelized, this drastically speeds up computations. An additional speed-up comes from the fact that optical computing occurs with the speed of light – which is, according to modern physics, the fastest possible process in the Universe.

Color representation beyond interval-valued fuzzy degrees. In describing colors, we may be interested not only in the color itself, but also in how bright or how light is this color. Interestingly, there is a natural way to extend our color-based fuzzy interpretation to this additional feature.

Namely, in 2011, Lotfi Zadeh proposed to make our description of the expert's opinion more adequate by adding,

- to the expert's degree of confidence – as described by a number from the interval $[0,1]$ or by a subinterval of this interval,
- an additional number that describes the expert's degree of confidence in this number.

Zadeh called the resulting pairs *Z-numbers*; see, e.g., [1, 6, 16].

For example, when asked whether a person who one saw for a second was tall, the witness may say "somewhat tall", and explain that he/she is somewhat – but not fully – confident about this. Thus, in this case, in addition to the numerical value corresponding to "somewhat tall", we have an additional number describing "somewhat confident".

This additional number – that described the degree to which the expert is confident about his/her statement – is a natural counterpart of the intensity of color. Thus, a color scheme can be naturally used not only to describe interval-valued fuzzy degrees, it can also be naturally used to describe interval-valued Z-numbers.

An additional confirmation that interval-valued fuzzy degrees are more adequate. An interesting side effect of our idea is that we now have a new argument of why interval-valued fuzzy degrees are more adequate than values from the interval $[0,1]$ – at least for describing sufficiently complex phenomena.

Indeed, as we have mentioned:

- while interval-valued degrees can be obtained as convex combinations of 3 basic degrees, and thus, correspond to combinations of 3 colors,
- $[0,1]$-based degrees are convex combinations of 2 basic degrees, and thus, correspond to combinations of 2 colors.

Herein lies the crucial difference:

- for general graphs, the question of whether a graph can be colored in 3 colors in known to be NP-hard,
- while the question of whether a graph can be colored in 2 color can be answered by a simple feasible algorithm;

see, e.g., [3]. Thus, if we want to be able to describe complex phenomena – for many of which important questions are NP-hard – we cannot use 2 colors, that would lead to models which are too simple, we need at least 3 colors. (And, by the way, 3 colors are sufficient to reflect the corresponding complexity.)

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