Logical Inference Inevitably Appears: Fuzzy-Based Explanation

Julio Urenda[a], Olga Kosheleva[b], Vladik Kreinovich[c], and Orsolya Csiszar[d]
[a]Math. Sciences, Univ. of Texas at El Paso, El Paso, TX 79968, USA, jcurenda@utep.edu
[b]Teacher Education, Univ. of Texas at El Paso, El Paso, TX 79968, USA, olgak@utep.edu
[c]Computer Science, Univ. of Texas at El Paso, El Paso, TX 79968, USA, vladik@utep.edu
[d]Aalen Univ. of Applied Sciences, Germany, orsolya.csiszar@hs-aalen.de

Abstract

Many thousands years ago, our primitive ancestors did not have the ability to reason logically and to perform logical inference. This ability appeared later. A natural question is: was this appearance inevitable – or was this a lucky incident that could have been missed? In this paper, we use fuzzy techniques to provide a possible answer to this question. Our answer is: yes, the appearance of logical inference in inevitable.

Keywords: Logical reasoning, Fuzzy logic, Historical emergence of logical reasoning, Schur's theorem

1 Main question: logical inference historically appeared, but was it inevitable?

Many thousands years ago, our primitive ancestors did not have the ability to reason logically and to perform logical inference. This ability appeared later. A natural question is:

- was this appearance inevitable,
- or was this a lucky incident that could have been missed?

In this paper, we use fuzzy techniques [1, 4, 5, 6, 7, 10] to provide a possible answer to this question. Our answer is: yes, the appearance of logical inference in inevitable.

2 Let us formulate this question in precise terms

Need to consider degrees of certainty. Nowadays, we know the statements which are absolutely true, namely, the statements of abstract mathematics. However, these statement already presuppose the ability to reason logically.

Since we are interested in analyzing how logical reasoning appeared in the first place, we need to ignore mathematical statements and concentrate on statements about the real world. In this case:

- if we go beyond observed facts – which are, of course, clearly true.
- such statements always come with some degree of certainty.

Indeed, we may observe some phenomenon many times, but it does not mean that we are 100% sure that this will always be true:

- Every day, we see the sun rising in the morning, but one day, there is a solstice, and the sun is not visible.
- Every day eating a certain plant is OK, but one day, a fungus attacks this plant, making it poisonous for humans, etc.

So, we need to deal with statements that have some degree of uncertainty.

We can combine these statements into complex ones. Once we have statements S_1, S_2, $\ldots$, we can combine them into logical combinations. For example, we can consider statements $S_1 \,\&\, S_2$, $S_3 \vee \neg S_4$, etc.

One of the main ideas behind fuzzy logic is that:

- if we know the degrees of certainty d_i in statements S_i,
- then we can estimate our degree of certainty in a combined statement by using the corresponding "and"-, "or"-, and "not"-operations $f_\&(a,b)$, $f_\vee(a,b)$, and $f_\neg(a,b)$.

Comment. For historical reasons:

- "and"-operations are usually known as *t-norms*, while
- "or"-operations are usually known as *t-conorms*.

End of comment.

So, if we consider the set D of degrees of certainty of all possible combined statements, this set must be closed under these operations, i.e.,

- if $a \in D$ and $b \in D$,
- then we must have $f_\&(a,b) \in D$, $f_\vee(a,b) \in D$, and $f_\neg(a) \in D$.

Let us restrict ourselves to intuitively reasonable "and"-operation. For non-mathematical statements, a combined statement "A and B" is, in general, stronger than each of the two statements A and B. So, it makes sense to consider "and"-operations that are consistent with this intuitive idea, i.e., for which:

- wherever $a < 1$ and $b < 1$,
- we have $f_\&(a,b) < a$ and $f_\&(a,b) < b$.

A person – or even a group – rarely deals with all possible degrees of certainty. Even now, it is rare that the same group of people deal with statements of all kinds degree of certainty. For example:

- mathematicians usually deal only with absolutely correct statements,
- physicists usually deal with statements that are correct on the physical level – i.e., have some uncertainty in them,
- biologists usually deal with statement that have even less degree of certainty,
- philosophers – unless they follow a formal approach – usually deal with statement with even less certainty, etc.

At each moment of time, there are several such groups of people. Let us denote the number of such groups by n. Let us denote by $D_1, \ldots, D_n$ the sets of degrees of certainty corresponding to each of these groups.

What does appearance of logical inference mean in these terms. In general, logical inference means that the same person – or at least the same group of people– deals both:

- with some statements, e.g., S_1 and S_2, and
- with their logical combination, e.g., $S_1 \& S_2$.

In these terms, the appearance of logical inference means that on some level, some logical combination of statement from this level also belongs to this same level.

Now, we are ready to formulate our result in precise terms.

Comment. To maintain the greatest possible degree of generality, we will use the weakest possible assumptions. For example:

- we will not assume that the degrees of certainty are numbers from the interval $[0,1]$; for example, we allow interval-values degrees of certainty (see, e.g., [5]), and
- we will not assume that the "and"-operation is commutative,

since these assumptions are not needed for our proof.

3 Definitions and the main result

Definition 1. *By* logical development*, we mean the tuple* $\langle D, f_\&, f_\vee, f_\neg, D_1, \ldots, D_n\rangle$*, where:*

- *D is a partially ordered set that contains the largest element 1 and also contains at least one element different from 1; its elements will be called* degrees of certainty*;*
- *$f_\& : D \times D \to D$ is an associative operation on D for which $f_\&(a,b) < a$ and $f_\&(a,b) < b$ whenever $a < 1$ and $b < 1$;*
- *$f_\vee : D \times D \to D$ and $f_\neg : D \to D$ are operations on D; and*
- *D_i are subsets of D for which $\cup D_i = D$.*

Definition 2. *We say that a value $d \in D$ is a* logical combination *of the values $d_1, \ldots, d_m \in D$ if d can be obtained from d_i by using at least one of the operations $f_\&(a,b)$, $f_\vee(a,b)$, and $f_\neg(a,b)$.*

Example. For example, we may have $d = f_\&(d_1, d_2)$, or $d = f_\vee(d_3, f_\neg(d_4))$, etc.

Definition 3. *We say that a logical development* contains logical reasoning *if one of the sets D_i contains both:*

- *some values $d_1, \ldots, d_m$, and*
- *a value d which is their logical combination.*

Proposition. *Every logical development contains logical reasoning.*

Discussion.

- This result means that as we consider more and more statements, eventually, there will be the case when some group will be dealing both:
  - with some statements *and*
  - with their logical combination,

  i.e., logical inference will indeed inevitably appear.
- The above proposition promised the existence of *some* logical combination. We will actually prove a more specific result: that on every logical development, there is a group D_i that contains both:
  - some elements d and d', and
  - their "and"-combination $f_\&(d,d')$.

Proof.

1°. Due to the first bullet item in Definition 1, the set D contains a degree d_1 which is smaller than 1. Let us consider, for each natural number $k > 1$, the degree d_k that is obtained by applying k times the "and"-operation $f_\&$ to d_1:

$$d_2 = f_\&(d_1,d_1),$$

$$d_3 = f_\&(d_2,d_1) = f_\&(f_\&(d_1,d_1),d_1),$$

$$d_4 = f_\&(d_3,d_1) = f_\&(f_\&(f_\&(d_1,d_1),d_1),d_1),$$

and, in general,

$$d_{k+1} = f_\&(d_k,d_1).$$

2°. By associativity, we can conclude that for all possible value k and ℓ, we have $f_\&(d_k,d_\ell) = d_{k+\ell}$.

3°. Since we have $f_\&(a,b) < a$ and $f_\&(a,b) < b$ whenever $a < 1$ and $b < 1$, we can prove, by induction, that the degrees d_k form a strictly decreasing sequence:

$$1 > d_1 > d_2 > \ldots > d_k > d_{k+1} > \ldots$$

This implies, in particular, that all the values d_k are different.

4°. Since $\cup D_i = D$, for each k, the degree d_k belongs to one of the groups D_i.

Let N_i denote the set of all the indices k for which $d_k \in D_i$. Then, we have $N = \cup N_i$.

5°. Now, we can use Schur's theorem (see, e.g., [2], p. 773), according to which:

- every time we divide the set of all natural numbers into finitely many subsets N_i,
- one of these subsets – let us denote it by N_j – contains integers k and ℓ for which the sum $k+\ell$ is also contained in this same subset.

Comment. Strictly speaking, Schur's theorem requires that we have a partition, and the sets N_i do not necessarily form a partition – some of them may have a non-empty intersection. However, this problem is easy to overcome if:

- instead of the original sets N_1, N_2, etc.,
- we consider sets $N'_1 = N_1$, $N'_2 = N_2 - N_1$,

  $$N'_3 = N_3 - (N_1 \cup N_2),$$

  and, in general,

  $$N'_i = N_i - (N_1 \cup \ldots \cup N_{i-1}).$$

Then, the sets N'_i form a partition. Thus, by Schur's Theorem, there exists a set N'_j that contains two numbers k, ℓ, and their sum $k+\ell$. Since $N'_j \subseteq N_j$, the original set N_j also contains these three numbers. *End of comment.*

By definition of the sets N_j, the fact that k, ℓ, and $k+\ell$ all belong to N_j means that

$$d_k \in D_j,\ d_\ell \in N_j, \text{ and } d_{k+\ell} \in D_j.$$

By Part 2 of this proof, this means that $f_\&(d_k,d_\ell) \in D_j$.

The proposition is thus proven.

Discussion. The above proposition says that for every n:

- if we continuously add degree of certainty so that eventually all degrees will be added,
- then, at some stage, we will reach a point at which logical reasoning emerges.

In this result, the point at which logical reasoning emerges may depend on the specific division of the set D into groups. However, there exists a stronger version of Schur's theorem according to which, for each n, there exists a number $N(n)$ for which:

- if we divide all the natural numbers from 1 to $N(n)$ into n groups $N_1, \ldots, N_n$,
- then one of these groups N_j contains some values k and ℓ for which $k + \ell \in N_j$.

In our terms, this means that:

- if we only consider degrees $d_1, \ldots, d_{N(n)}$,
- then among these degrees, one of the groups D_j will contain elements d_k, d_ℓ, and

$$d_{k+\ell} = f_\&(d_k, d_\ell).$$

A slightly stronger result. Another generalization of the original Schur's theorem is Folkman's theorem ([3], pp. 65–69; see also [8, 9]), according to which:

- for each division of the set of natural numbers N into a finite number of subsets N_i, and for each $m > 1$,
- there exists a subset N_j and m elements from this subset for which the sum of any number of them is still in N_j.

In our terms, this means that:

- not only we have two degrees $d_k, d_\ell \in D_j$ for which $f_\&(d_k, d_\ell) \in D_j$, but
- we also have m elements $d_{k_1}, \ldots, d_{k_m} \in D_j$ for which any "and"-combination $f_\&(d_{k_{j(1)}}, d_{k_{j(2)}}, \ldots)$ also belongs to D_j.

In other words:

- not only the simplest form of logical inference eventually appear, but also
- more and more sophisticated versions of logical reasoning eventually appear.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395, and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

The authors are thankful to the anonymous referees for valuable suggestions.

References

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] W. R. Bloom, R. J. Gardner, A. Hales, J. Spencer, T. Tao, and B. Weiss, "Robert Israel 'Bob' Jewett (1937–2022)", *Notices of the American Mathematical Society*, 2023, Vol. 70, No. 5, pp. 772–781.

[3] R. L. Graham, B. L. Rotschild, and J. H. Spencer, *Ramsey Theory*, Wiley, Nee York, 1980.

[4] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[5] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[6] H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.

[7] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[8] R. Rado, "Some partition theorems", In: *Combinatorial Theory and Its Applications, Proceedings of the III Colloquium, Balatonfüred, 1969*, North-Holland, Amsterdam, 1969, pp. 929–936.

[9] J. H. Sanders, *A Generalization of Schur's Theorem*, PhD Dissertation, Yale University, 1968.

[10] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.