How to Make Decision Under Interval Uncertainty: Description of All Reasonable Partial Orders on the Set of All Intervals

Tiago M. Costa[a], **Olga Kosheleva**[b], and **Vladik Kreinovich**[c]
[a] Dept. Matemática, Univ. de Tarapacá, Casilla 7D, Arica, Chile, grafunjo@yahoo.com.br
[b] Teacher Education, Univ. of Texas at El Paso, El Paso, TX 79968, USA, olgak@utep.edu
[c] Computer Science, Univ. of Texas at El Paso, El Paso, TX 79968, USA, vladik@utep.edu

Abstract

In many practical situations, we need to make a decision while for each alternative, we only know the corresponding value of the objective function with interval uncertainty. To help a decision maker in this situation, we need to know the (in general, partial) order on the set of all intervals that corresponds to the preferences of the decision maker. For this purpose, in this paper, we provide a description of all such partial orders – under some reasonable conditions. It turns out that each such order is characterized by two linear inequalities relating the endpoints of the corresponding intervals, and that all such orders form a 2-parametric family.

Keywords: decision making, interval uncertainty, partial order, decision making under interval uncertainty.

1 Formulation of the Problem

Need to make decisions under uncertainty. In many practical situations, we have several alternatives to select from, and we have an objective function that described our preferences. For example, when we design an electric car:

- we may want to maximize the distance that it can run until the next charge, or
- we can minimize its weight, etc.

In the ideal case, when for each alternative, we know the exact value of the objective function. In this case:

- If we want to maximize the objective function, we select the alternative with the largest value of this function.
- If we want to minimize the objective function, we select the alternative with the smallest value of this function.

However, in practice, we rarely know these exact values. What we usually know instead is the *interval* of possible values. It is therefore necessary to make a decision based on these interval values; see, e.g., [3, 6, 8, 10, 13].

Comment. In this paper, we analyze the problem of decision making under such interval uncertainty. This problem – as we will show – is already not easy.

In practice, the situation is sometimes even more complex than that. For example:

- in addition to the interval of possible values,
- we may have experts describing us to what extent each of these values is possible.

In this case, each alternative is characterized not just by an interval, but, in effect, by a *fuzzy set* of possible values; see, e.g., [1, 4, 9, 11, 12, 15].

We hope that our analysis of decision making in the case of interval uncertainty can help to come up with similar techniques for decision making in this more complex case of fuzzy uncertainty.

Why decision making under interval uncertainty is not easy. Let us explain why decision making under interval uncertainty is not easy. For example:

- If we want to maximize the value of the objective function, then the value 4 is clearly better than the value 3.
- However, it is not clear whether, e.g., the interval $[2,5]$ corresponding to one alternative is better that the interval $[3,4]$ corresponding to another alternative.

Sometimes, a decision maker may have a clear preference between the two intervals. In other cases, the decision maker may be undecided. In other words:

- in contrast to comparing real numbers, where we have a linear (total) order, i.e., where for every two different numbers either the first one is larger or the second one is larger,
- for comparing intervals, in general, we have *partial* order: sometimes one interval is better than another interval, and sometimes, there is no relation between them.

So, to help decision makers make good decisions in such situations, we need to know the partial order between intervals that describes the decision maker's preferences.

What we do in this paper. In this paper, we describe all possible partial orders that satisfy some reasonable properties.

2 What Are the Reasonable Properties of a Partial Order

What we do in this section. Let us first analyze what are reasonable properties that the partial order $\le$ on the set of all intervals must satisfy. To make our analysis clearer and more convincing, we will illustrate these properties not on complex examples like electric car design, but on simple financial examples. In this case:

- the numerical value is a monetary gain, and
- an interval means that we are not sure what will be the monetary gain in this situation, we only know the range of possible values of this gain.

First reasonable property: additivity. Suppose that the decision maker needs to decide between two alternatives characterized by the intervals $\mathbf{a} = [\underline{a}, \overline{a}]$ and $\mathbf{b} = [\underline{b}, \overline{b}]$.

An important point is that gains rarely come from only one source. In addition to the gain that will come from the decision maker selecting one of these two alternatives, this decision maker may be bound to receive some additional amount resulting from his/her previous decisions. This additional amount may also not be known exactly, we may only know the interval $\mathbf{c} = [\underline{c}, \overline{c}]$ of possible gains.

With this additional gain in mind, we have two different ways to look at the original choice problem. We can ignore this additional gain and consider it as the problem of selecting between the intervals $\mathbf{a}$ and $\mathbf{b}$. Alternatively, we can consider the overall gains of the decision maker in this situation.

- If the decision maker selects the alternative $\mathbf{a}$, then the possible values of his/her overall gain form an interval
$$\mathbf{a} + \mathbf{c} = \{a + c : a \in \mathbf{a} \text{ and } c \in \mathbf{c}\}.$$
The smallest possible value of this sum is attained when both values a and c are the smallest, and the largest possible value of this overall gain is attained when both values a and c are the largest. Thus, this interval is equal to
$$\mathbf{a} + \mathbf{c} = [\underline{a} + \underline{c}, \overline{a} + \overline{c}].$$
- Similarly, if the decision maker selects the alternative $\mathbf{b}$, then the possible values of his/her overall gain form an interval
$$\mathbf{a} + \mathbf{c} = [\underline{b} + \underline{c}, \overline{b} + \overline{c}].$$

In this alternative description, we need to select between the two intervals $\mathbf{a} + \mathbf{c}$ and $\mathbf{b} + \mathbf{c}$.

These are two different descriptions of the exact same decision situation. So, it makes sense to require that the decision maker selects $\mathbf{b}$ over $\mathbf{a}$ if and only if he/she selects $\mathbf{b} + \mathbf{c}$ over $\mathbf{a} + \mathbf{c}$. Thus, we arrive at the first reasonable requirement.

Definition 1. *A partial order $\le$ on the set of all intervals is called* additive *if for every three intervals* $\mathbf{a}$, $\mathbf{b}$, *and* $\mathbf{c}$, *we have:*
$$\mathbf{a} \le \mathbf{b} \Leftrightarrow \mathbf{a} + \mathbf{c} \le \mathbf{b} + \mathbf{c}.$$

Second reasonable property: antisymmetry. Sometimes, we have zero-sum situations in which one party's gain is another party's loss. Let us consider the simplest case when the two parties has the same preferences. In this case:

- if for the first party, the gain a is better than the gain b,
- this would mean that for the second party, the loss of a should be worse than the loss of b.

The loss of a is, in mathematical terms, the same as gain $-a$. Thus, the requirement is that if $a \le b$, then we should have $-b \le -a$.

The same should be true for interval-valued gains, where for each interval $\mathbf{a} = [\underline{a}, \overline{a}]$, we have
$$-\mathbf{a} \stackrel{\text{def}}{=} \{-a : a \in \mathbf{a}\} = [-\overline{a}, -\underline{a}].$$

Thus, we arrive at the following definition:

Definition 2. *A partial order $\le$ on the set of all intervals if called* antisymmetric *if for every two intervals* **a** *and* **b**, *we have*

$$\mathbf{a} \le \mathbf{b} \Rightarrow -\mathbf{b} \le -\mathbf{a}.$$

Comment. For real numbers, a similar property does not need to be separately formulate, since for real numbers, this property follows from additivity. Indeed, if $a \le b$, then for $c = -a - b$, additivity implies that

$$a + (-a - b) \le b + (-a - b),$$

i.e., that $-b \le -a$.

This conclusion is based on the fact that for real numbers, $a + (-a) = 0$. However, this derivation cannot be applied to intervals, since for intervals, in general, the sum $\mathbf{a} + (-\mathbf{a})$ is different from 0: e.g.,

$$[0,1] + (-[0,1]) = [0,1] + [-1,0] =$$

$$[0 + (-1), 1 + 0] = [-1,1] \ne 0.$$

Third reasonable property: homogeneity. In general, if the gain b is better than the gain a, then:

- half of b is still better than half of a,
- twice the gain of b is better than twice the gain of a, and,
- in general, for every $\lambda > 0$, $\lambda \cdot b$ is better than $\lambda \cdot a$.

It is reasonable to require the same property for intervals, where $\lambda \cdot \mathbf{a}$ means

$$\lambda \cdot [\underline{a}, \overline{a}] \stackrel{\text{def}}{=} \{\lambda \cdot a : a \in [\underline{a}, \overline{a}]\}.$$

By using the same monotonicity argument as in the case of addition, one can check that this interval is equal to

$$\lambda \cdot [\underline{a}, \overline{a}] = [\lambda \cdot \underline{a}, \lambda \cdot \overline{a}].$$

Thus, we arrive at the third reasonable property.

Definition 3. *A partial order $\le$ on the set of all intervals if called* homogeneous *if for every two intervals* **a***a and* **b** *an for every real number $\lambda > 0$, we have*

$$\mathbf{a} \le \mathbf{b} \Rightarrow \lambda \cdot \mathbf{a} \le \lambda \cdot \mathbf{b}.$$

Now, we are ready to formulate our main result.

3 Main Result: Formulation and Proof

Proposition. *For every partial order $\le$ on the set of intervals, the following two conditions are equivalent to each other:*

- *the order is additive, antisymmetric, and homogeneous;*
- *there exist real numbers $\underline{c}_1$, $\overline{c}_1$, $\underline{c}_2$, $\overline{c}_2$ and relations $<_1, <_2 \in \{<, \le\}$ for which*

$$[\underline{a}, \overline{a}] \le [\underline{b}, \overline{b}] \Leftrightarrow$$

$$\underline{c}_1 \cdot \underline{a} + \overline{c}_1 \cdot \overline{a} <_1 \underline{c}_1 \cdot \underline{b} + \overline{c}_1 \cdot \overline{b} \text{ and}$$

$$\underline{c}_2 \cdot \underline{a} + \overline{c}_2 \cdot \overline{a} <_1 \underline{c}_2 \cdot \underline{b} + \overline{c}_2 \cdot \overline{b}.$$

Comments.

- At first glance, it may look like we need 4 parameters $\underline{c}_i$ and $\overline{c}_i$ to describe a general partial order. However, it is easy to see that we only need two parameters: in each pair $(\underline{c}_i, \overline{c}_i)$, we can divide both sides by each inequality the absolute value of a non-zero parameter and thus, get an equivalent inequality with only one parameter.
- In the particular case of linear (total) order, we conclude that the decision means selecting an alternative with the largest (of smallest) value of a linear combination of lower and upper endpoints of the interval. This is, in effect, what is known as Hurwicz optimism-pessimism criterion, a criterion that was awards by a Nobel prize [2, 5, 7].

Proof.

1°. It is easy to check that every partial order than is described by the parameters $\underline{c}_i$ and $\overline{c}_i$ is additive, antisymmetric, and homogeneous.

So, to complete the proof, it is sufficient to prove that every additive, antisymmetric, and homogeneous partial order $\le$ has this form. In the following proof, we will assume that the order $\le$ has these three properties, and we will show that it has the desired form.

2°. Let us first prove that the order $\le$ is uniquely determined by the set C of all the interval **a** for which $[0,0] \le \mathbf{a}$. For this purpose, we will consider two possible cases:

- the case when the width $\overline{b} - \underline{b}$ of the interval **b** is larger than or equal to the width $\overline{a} - \underline{a}$ of the interval **a**, and
- the case when the interval **a** has the larger width.

2.1°. In the first case, by adding $\underline{b}-\overline{a}$ to both sides of the inequality $\overline{b}-\underline{b} \geq \overline{a}-\underline{a}$, we conclude that

$$\overline{b}-\overline{a} \geq \underline{b}-\underline{a}.$$

Thus, we can have an interval $[\underline{b}-\underline{a},\overline{b}-\overline{a}]$. We will denote this interval by $\mathbf{b}\ominus\mathbf{a}$.

One can easily check that we have $\mathbf{b}=(\mathbf{b}\ominus\mathbf{a})+\mathbf{a}$ and that we have $\mathbf{a}=[0,0]+\mathbf{a}$. Thus, by additivity, we have

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow [0,0] \leq \mathbf{b}\ominus\mathbf{a},$$

i.e., by definition of the set C:

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow \mathbf{b}\ominus\mathbf{a} \in C.$$

2.2°. In the second case, by adding $\underline{a}-\overline{b}$ to both sides of the inequality $\overline{b}-\underline{b} < \overline{a}-\underline{a}$, we conclude that

$$\underline{a}-\overline{b} < \overline{a}-\overline{b}.$$

Thus, we can have an interval $\mathbf{a}\ominus\mathbf{b}$. One can easily check that we have $\mathbf{a}=(\mathbf{a}\ominus\mathbf{b})+\mathbf{b}$ and that we have $\mathbf{b}=[0,0]+\mathbf{b}$. Thus, by additivity, we have

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow \mathbf{a}\ominus\mathbf{b} \leq [0,0].$$

Now, due to antisymmetry, the condition $\mathbf{a}\ominus\mathbf{b} \leq [0,0]$ is equivalent to

$$[0,0] \leq -(\mathbf{a}\ominus\mathbf{b}) = [\overline{b}-\overline{a},\underline{b}-\underline{a}],$$

i.e.,

$$\mathbf{a} \leq \mathbf{b} \Leftrightarrow [\overline{b}-\overline{a},\underline{b}-\underline{a}] \in C.$$

Thus, indeed, the order $\leq$ is uniquely determined by the set C.

3°. Let us use a natural geometric representation of intervals to analyze the properties of the set C.

Namely, each interval $[\underline{a},\overline{a}]$ can be naturally represented by a planar point with coordinates $(\underline{a},\overline{a})$. In this representation, the set C becomes a set of such points, i.e., a planar set.

3.1°. Due to homogeneity, if $[0,0] \leq \mathbf{a}$, then for each $\lambda > 0$, we have $[0,0] \leq \lambda\cdot\mathbf{a}$.

In geometric terms, this means that the set C is closed under multiplication by a positive number.

3.2°. If the set C contains two intervals $\mathbf{a}$ and $\mathbf{b}$, i.e., if $[0,0] \leq \mathbf{a}$ and $[0,0] \leq \mathbf{b}$ then by additivity, we get $[0,0]+\mathbf{b} \leq \mathbf{a}+\mathbf{b}$, i.e., $\mathbf{b} \leq \mathbf{a}+\mathbf{b}$. So, since partial order is transitive, we et $[0,0] \leq \mathbf{a}+\mathbf{b}$.

In geometric terms, this means that the set C is closed under addition.

3.3°. Thus, the set C is closed under multiplication by a positive constant and under addition.

This means that the set C is a convex cone, and it is known that all convex cones in a plane have the desired representation: namely, they form a space between two lines – each of which is described by a homogeneous linear equation; see, e.g., [14].

The proposition is thus proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395, and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

The authors are thankful to all the participants of the 2023 Annual Conference on the North American Fuzzy Information Processing Society (Cincinnati, Ohio, May 31 – June 2, 2023) for valuable discussions.

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