Methodological Lesson of Pythagorean Triples

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Abstract There are many right triangles in which all three sides a, b, and c have integer lengths. The triples (a,b,c) formed by such lengths are known as Pythagorean triples. Since ancient times, it is known how to generate all Pythagorean triples: we can enumerate primitive Pythagorean triples – in which the three numbers have no common divisors – by considering all pairs of natural numbers $m > n$ in which m and n have no common divisors, and taking $a = m^2 - n^2$, $b = 2m \cdot n$, and $c = m^2 + n^2$. Multiplying all elements of a triple by the same number, we can get all other Pythagorean triples. The proof of this result – going back to Euclid – is technical. In this paper, we provide a commonsense explanation of this result. We hope that this explanation – which is more general than Pythagorean triples – can lead to new hypotheses and new results about similar situations.

1 Formulation of the Problem

Pythagorean triples: what are they and why they are interesting. According to Pythagoras Theorem, for a right triangle, the square of the hypotenuse c is equal to the sum of the squares of its sides a and b:

$$c^2 = a^2 + b^2. \tag{1}$$

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In some cases, we can have both a, b, and c as positive integers. The simplest example is $a=3$, $b=4$, and $c=5$. Because of their relation to Pythagoras Theorem, such triples (a,b,c) are known as *Pythagorean triples*. However, many of them were known way before Pythagoras proved this result. For example, the triple $(3,4,5)$ was used in the ancient world – e.g., by the ancient Babylonians – to design the corresponding right triangle and thus, to get a perfect right angle.

How can we generate all Pythagorean triples: Euclid's formula. Already ancient Greeks knew how to generate all Pythagorean triples; see, e.g., Proposition XXIX, Book X of Euclid's Element [3]. Namely, it is clear that if (a,b,c) is a Pythagorean triple, then for any integer $k>1$, the triple $(k\cdot a, k\cdot b, k\cdot c)$ also satisfies the formula (1). Thus, to describe all Pythagorean triples, it is sufficient to describe all *primitive* triples, i.e., triples for which the three numbers a, b, and c have no common divisor.

Euclid's formula describes the general form of primitive Pythagorean triples. They all come from selecting two integers $m>n$ that have no common divisors. Once we select m and n, we can then take:

$$a=m^2-n^2,\ \ b=2m\cdot n,\ \text{and}\ c=m^2+n^2.$$

Why do we need a methodological lesson here? At first glance, the above formula that describes all primitive Pythagorean triples is very precise and very clear, so why do we need to muddle it with methodology?

The reason for this is straightforward. The above result is a very technical result. It is known, however, that often, in mathematics, results lead to new results – often not purely by technical arguments, but by first formulating the original result in an imprecise form – what we call, somewhat imprecisely (pun intended), methodological form. Once the original result is described in an imprecise form – stripped of technical details – it automatically becomes easier to generalize. Then, some specific cases of thus generalized statement lead to new results – new hypotheses and eventually new theorems.

What we do in this paper. In this paper, we follow the above pattern, at least its first part: namely, we reformulate Euclid's result about primitive Pythagorean triples in more general imprecise terms. We hope that this reformulation will inspire readers to come up with new hypotheses and, hopefully, new interesting results.

2 So What Is a Methodological Lesson of Pythagorean Triples?

Where else sums of squares appear. In other to come up with the desired generalization, let us recall where else in mathematics sums of squares naturally appear.

What naturally comes to mind is that in many ordered algebraic structures – e.g., in ordered fields – an element is positive if and only if it can be represented as a sum of non-zero squares. To be more precise, since every non-zero square is positive and the sum of positive elements is positive. Such sums of squares form the smallest possible set of positive elements. This corresponds to a natural order in many fields; see, e.g., [4].

Natural idea. The above example prompts a natural classification of positive elements by the complexity of their representation:

- the simplest are numbers that are non-zero squares, i.e., numbers of the form c^2;
- next in simplicity are numbers that can be represented as the sum of two non-zero squares, i.e., numbers of the form a^2+b^2;
- the third level of complexity is occupied by numbers that can be represented as the sum of three non-zero squares $a^2+b^2+c^2$, etc.

In particular, for natural numbers, we only have four levels of this hierarchy, since (as proven by Lagrange in 1770), every positive integer can be represented as a sum of at most four non-zero squares; see, e.g., [2].

Can we have a situation which is simpler than the simplest? At first glance, the above classes is all we have, and there seems to be no way to have something simpler than the simplest class. But it actually *is* possible: when a number can be represented *both* as a non-zero square *and* as the sum of two non-zero squares.

This is exactly the case of the numbers $n = c^2$ from Pythagorean triples: each of these numbers n can be represented both as a non-zero square c^2 and as the sum a^2+b^2 of two non-zero squares.

Let us describe this situation in general terms. We have a system (in our case, a triple) which is unusually simple. What can we conclude about it?

Well, we may not have that much intuition about simplicity, so let us consider a similar setting for a property for which we have a better intuition – e.g., weight. In general, the more parts a system has, the more it weighs. But what is a system with a few parts is very heavy? That would mean that one of its components is heavier than average. Similarly, if a system is unusually light – as compared to what we would expect based on its number of components – this means that at least some of these components are lighter than average.

Similarly, it is reasonable to expect that when a system is simpler than average, this probably means that one of its components is simpler than average.

How does this general description apply to the Euclid's result. We have a number n that is unusually simple in terms of the above classification, because it can be represented both as a non-zero square c^2 and as the sum of two non-zero squares a^2+b^2. The above methodological idea implies that one of its "components" should be simpler than average.

What are components here? We have two representations for the number n. Out of these two, the main representation – corresponding to the greater simplicity – is the representation as c^2, i.e., a product $c \cdot c$. From this viewpoint, a natural idea is to view these two equal factors c as components of n.

So, the above general methodological idea seems to indicate that for Pythagorean triples, the number c should be much simpler than average. In our discussion, "simple" means "is a sum of few non-zero squares". It is known:

- that all numbers can be represented as sums of at most four squares,
- that the vast majority of numbers *can* be represented as sums of three squares: the only exceptions are numbers of the type $4^m \cdot (8k+7)$; and
- that the vast majority of numbers are *not* squares and *cannot* be represented as sums of two non-zero squares.

From this viewpoint, the average simplicity means that a number can be represented as a sum of three non-zero squares. Thus, the conclusion that the component c is simpler than average means that this number can be represented as a sum of fewer than three non-zero squares, i.e., that:

- either c is itself a square,
- or c is the sum of two non-zero squares.

To be on the cautious size:

- if a system is heavier than expected, this means at least one of its components is heavier than average – but we should not necessarily conclude that it is *much* heavier;
- similarly, if a system is simpler than expected, this means that at least one of its components is simpler than average – but we should not necessarily conclude that it is *much* simpler.

So, it is reasonable to dismiss the case when c is itself a square – that would mean that it is much simpler than average. Thus, we conclude that c should be the sum of two non-zero squares.

And this is exactly what Euclid's result says: that the number c itself can be represented as the sum m^2+n^2 of the two squares.

So what? Well, we hopefully got some intuitive understanding of Euclid's result – beyond technical manipulations, and we also got a reasonable general principle. This principle could be helpful: if we did not know how to describe all Pythagorean triples, it would help. Thus, we hope that it can be helpful in coming up with new hypotheses – maybe about ordered fields, maybe about something completely different.

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