We Can Always Reduce a Non-Linear Dynamical System to Linear – at Least Locally – But Does It Help?

Orsolya Csiszár, Gábor Csiszár, Olga Kosheleva, Vladik Kreinovich, and Nguyen Hoang Phuong

Abstract Many real-life phenomena are described by dynamical systems. Sometimes, these dynamical systems are linear. For such systems, solutions are well known. In some cases, it is possible to transform a nonlinear system into a linear one by appropriately transforming its variables, and this helps to solve the original nonlinear system. For other nonlinear systems – even for the simplest ones – such transformation is not known. A natural question is: which nonlinear systems allow such transformations? In this paper, we show that we can always reduce a nonlinear system to a linear one – but, in general, it does not help, since the complexity of

---

Orsolya Csiszár
Research Center for Complex Systems, Aalen University
Beethovenstrasse 1, 73431 Aalen, Germany
e-mail: orsolya.csiszar@hs-aalen.de
and Institute of Applied Mathematics
John von Neumann Faculty of Informatics
Óbuda University, Bécsi Street 96/B, Budapest H-1034, Hungary
e-mail: csiszar.orsolya@nik.uni-obuda.hu

Gábor Csiszár
Biomatics and Applied AI Institute
John von Neumann Faculty of Informatics
Óbuda University, Bécsi Street 96/B, Budapest H-1034, Hungary
e-mail: csiszar.gabor@uni-obuda.hu

Olga Kosheleva
Department of Teacher Education, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: olgak@utep.edu

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: vladik@utep.edu

Nguyen Hoang Phuong
Artificial Intelligence Division, Information Technology Faculty, Thang Long University
Nghiem Xuan Yem Road, Hoang Mai District, Hanoi, Vietnam
e-mail: nhphuong2008@gmail.com

finding such a reduction is exactly the same as the complexity of solving the original nonlinear system.

1 Formulation of the problem

Dynamical systems are ubiquitous. One of the main objectives of science is to predict the future state of different systems. The state of a system can be described by the values $x_1,\ldots,x_n$ of different quantities that characterize this system. In many cases, the current state of the system uniquely determines the rate $\dot{x}_i$ with which each of the values changes:

$$\dot{x}_i = f_i(x_1,\ldots,x_n),\ \ 1 \le i \le n. \tag{1}$$

Such a system of differential equations is known as a *dynamical system*.

For a dynamical system, the current state of the system uniquely determines its state at any future (or past) moment of time; see, e.g., [1, 2].

In many cases, we have linear dynamical systems. In many practical situations, the changes are relatively small. In such cases, for all moments of time, the values x_i are close to their initial values $x_i^{(0)}$. In such cases, it is convenient to describe the state of the system by the differences $y_i \stackrel{\text{def}}{=} x_i - x_i^{(0)}$ for which $x_i = x_i^{(0)} + y_i$ and, thus, $\dot{x}_i = \dot{y}_i$. In terms of these differences, the equations (1) take the form

$$\dot{y}_i = f\left(x_1^{(0)} + y_1,\ldots,x_n^{(0)} + y_n\right). \tag{2}$$

Since the values y_i are relatively small, we can safely ignore terms that are quadratic or of higher order in terms of y_i. For example, if $y_i \approx 10\%$, then $y_i^2 \approx 1\%$, which is much smaller than y_i. Thus, we can expand the right-hand side of the formula (2) in Taylor series in terms of y_i and keep only linear terms in this expansion. In this case, the right-hand side of the system (2) becomes linear in $y_1,\ldots,y_n$;

$$\dot{y}_i = a_i + \sum_{j=1}^{n} a_{i,j} \cdot y_j, \tag{3}$$

for some coefficients a_i and $a_{i,j}$. Such dynamical systems are called *linear*.

Linear systems are easy to solve. There are known formulas for solving linear systems.

Namely, in the absence of the constant terms a_i, the general solution to this system is a linear combination of the terms $t^k \cdot \exp(\lambda \cdot t)$, where:

- λ is an eigenvalue of the matrix $a_{i,j}$, and
- k is a non-negative integer which is smaller than the multiplicity of this eigenvalue.

A simple modification of this formula – in most cases, by adding constants – takes care of the general case of (3).

Sometimes, we can reduce a non-linear system to a linear one. In some cases, the differences y_i are not small, so we have to consider the original nonlinear dynamical system (1). For such systems, there is no general way to solve it other than to solve it numerically. However, in some cases, it is possible to reduce a nonlinear system to a linear one and thus, to get an explicit solution.

A known example – for which such a reduction is possible – is the following system:

$$\dot{x}_1 = a \cdot x_1, \tag{4}$$

$$\dot{x}_2 = b \cdot (x_2 - x_1^2). \tag{5}$$

In this system, the first equation (4) is linear, but the second (5) is nonlinear.

This system can be reduced to a system of linear equations if we add the third variable $x_3 = x_1^2$. In this case, the second equation (5) becomes linear:

$$\dot{x}_2 = b \cdot x_2 - b \cdot x_3, \tag{6}$$

and for the derivative of x_3, we have

$$\dot{x}_3 = \frac{d}{dt}(x_1^2) = 2 \cdot x_1 \cdot \dot{x}_1.$$

Substituting the expression (4) for $\dot{x}_1$ into this formula, we get

$$\dot{x}_3 = 2 \cdot x_1 \cdot (b \cdot x_1) = 2b \cdot x_1^2.$$

By definition of x_3 as x_1^2, this implies that for the rate of change of the third variables, we also have a linear equation

$$\dot{x}_3 = 2b \cdot x_3. \tag{7}$$

For other nonlinear equations, such a reduction is not known. Let us consider the simplest possible nonlinear dynamical system.

- The more variables, the more complex the system, so the simplest system should have a single variable x_1 – and thus, the dynamical system consists of a single differential equation $\dot{x}_1 = f_1(x_1)$. To find the simplest of such systems, we need to select the simplest possible nonlinear function $f_1(x_1)$.
- The simplest possible functions are linear, and the simplest nonlinear functions are quadratic. The simplest-to-compute quadratic function is simply the expression x_1^2 that can be computed by a single multiplication. So, the simplest possible nonlinear system is the equation

$$\dot{x}_1 = x_1^2. \tag{8}$$

For this system – as well as for other differential equations with quadratic right-hand sides – no reduction to linear systems is known.

Natural questions. For some nonlinear systems, there is a known reduction to linear ones, and this reduction helps solve the system. This fact leads to the following two natural questions:

- which nonlinear systems can be reduced to linear ones? and
- if such a reduction is possible, can it help solve the original nonlinear system?

What we do in this paper. In this paper, we provide answers to both questions. Namely, we show:

- that *every* nonlinear system can be reduced to a linear one – at least locally, but
- that this does not help us solve the original nonlinear system – finding this reduction is as complicated as actually solving the system.

2 Our Answers

Notation. As we have mentioned, in general:

- once we know the equations (1) and we know the state $x = (x_1, \ldots, x_n)$ of the system at some moment t_0,
- we can uniquely determine its state at any other moment of time (at least locally, i.e., for times t sufficiently close to t_0).

Since the system (1) does not explicitly include time, the transition between the state at moment t_0 and the state at moment t depends only on the difference $t - t_0$. Let us denote the state at moment t corresponding to the state x at moment t_0 by $T_{t-t_0}(x)$. The components of the state $T_{t-t_0}(x)$ will be denoted by $T_{t-t_0,1}(x), \ldots, T_{t-t_0,n}(x)$. In these terms, the state $x = (x_1, \ldots, x_n)$ is transformed into the state

$$(z_1, \ldots, z_n) = (T_{t-t_0,1}(x_1, \ldots, x_n), \ldots, T_{t-t_0,n}(x_1, \ldots, x_n)).$$

Clearly, if we let the system evolve for time t and then again for time t', this is equivalent to letting it evolve for time $t + t'$, i.e., we have

$$T_{t+t'}(x) = T_{t'}(T_t(x)). \tag{8}$$

How to reduce a nonlinear system to linear: idea. Let us consider the cases when the values of the last variable x_n are close to some number $x_n^{(0)}$. In general, unless the system is degenerate and the value x_n does not change at all, the value x_n changes with time. For each tuple $(x_1, \ldots, x_{n-1}, x_n)$, we can look for the time τ at which the value x_n is equal to $x_n^{(0)}$, i.e., at which

$$T_{\tau,n}(x_1,\ldots,x_n) = x_n^{(0)}. \tag{9}$$

To find the unknown time τ, we have one equation (9) with one unknown τ. In general, if the number of equations is equal to the number of unknowns, the system has a unique solution – at least locally. Thus, we can always find such time τ.

Towards the resulting reduction. Now, instead of the original variables $x_1,\ldots,x_n$, we can use the new variables $(y_1,\ldots,y_n)$ in which:

- the value y_n is equal to the time τ determined by the formula (9), and
- the values $y_1,\ldots,y_i,\ldots,y_{n-1}$ are equal to

$$y_i = T_{\tau,i}(x_1,\ldots,x_n). \tag{10}$$

How will the variables y_i change with time? Due to formulas (9) and (10), we have

$$\left(y_1,\ldots,y_{n-1},x_n^{(0)}\right) = T_\tau(x_1,\ldots,x_n),$$

i.e., since $\tau = y_n$, we have

$$\left(y_1,\ldots,y_{n-1},x_n^{(0)}\right) = T_{y_n}(x_1,\ldots,x_n). \tag{11}$$

Thus, to describe x in terms of y, it is sufficient to trace the changes of the state $\left(y_1,\ldots,y_{n-1},x_n^{(0)}\right)$ back in time for the period $\tau = y_n$:

$$(x_1,\ldots,x_n) = T_{-y_n}\left(y_1,\ldots,y_{n-1},x_n^{(0)}\right). \tag{12}$$

In time t, the state x turns into $T_t(x)$. By applying the transformation T_t to both sides of the formula (11) and taking into account the formula (8), we get

$$T_t(x_1,\ldots,x_n) = T_{-(y_n-t)}\left(y_1,\ldots,y_{n-1},x_n^{(0)}\right). \tag{13}$$

Thus, by definition of the y-state, the y-state corresponding to $T_t(x)$ has the form $(y_1,\ldots,y_{n-1},y_n-t)$. Thus, with time:

- the new variables $y_1,\ldots,y_{n-1}$ do not change at all, and
- the variable y_n linearly decreases with time.

So, we indeed have the desired reduction. Let us summarize it.

Resulting reduction. For each state $x = (x_1,\ldots,x_n)$, let us define the new variables $y = (y_1,\ldots,y_n)$ as follows: y_n is the solution to the equation

$$T_{y_n,n}(x_1,\ldots,x_n) = x_n^{(0)}, \tag{14}$$

and for all $i < n$, we have

$$y_i = T_{y_n,i}(x_1,\ldots,x_n). \tag{15}$$

For the new variables, the dynamical system has the following form:

$$\dot{y}_1 = \ldots = \dot{y}_{n-1} = 0, \;\; \dot{y}_n = -1. \tag{16}$$

From the new variables y_i, we can get back to the original variables x_i by using the formula (12).

Example: derivation. Let us illustrate our reduction on the example of the simplest nonlinear dynamical systems $\dot{x}_1 = x_1^2$. This system is easy to solve. Indeed, we start with the original equation:

$$\frac{dx_1}{dt} = x_1^2.$$

We then separate the variables by multiplying both sides by dt and dividing both sides by x_1^2. This results in:

$$\frac{dx_1}{x_1^2} = dt.$$

If we integrate both sides of this equation, we get

$$-\frac{1}{x_1} + C = t,$$

where C denotes the integration constant. Thus,

$$\frac{1}{x_1} = C - t,$$

and so,

$$x_1(t) = \frac{1}{C-t}. \tag{17}$$

Let us describe the corresponding function $T_t(x_1)$ that describes the transition from the state at moment 0 to the state at moment t. At moment 0, the formula (17) leads to

$$x_1(0) = \frac{1}{C},$$

so

$$C = \frac{1}{x_1(0)}.$$

Substituting this value C into the formula (17), we conclude that

$$x_1(t) = \frac{1}{\dfrac{1}{x_1(0)} - t} = \frac{x_1(0)}{1 - x_1(0) \cdot t}.$$

Thus,

$$T_t(x_1) = \frac{x_1}{1 - x_1 \cdot t}. \tag{18}$$

Let us take $x_1^{(0)} = 1$. Then, the value y_1 should be determined by the equation (14) which, in this case, has the form

$$\frac{x_1}{1 - x_1 \cdot y_1} = 1,$$

hence $x_1 = 1 - x_1 \cdot y_1$, so $x_1 \cdot y_1 = 1 - x_1$, and

$$y_1 = \frac{1 - x_1}{x_1} = \frac{1}{x_1} - 1. \tag{19}$$

For the new variable y_1, the dynamical system takes a linear form: $\dot{y}_1 = -1$.

Once we know new state y_1, we can reconstruct the original state x_1 by using the formula (12), which in this case takes the form

$$x_1 = T_{-y_1}(1) = \frac{1}{1 - 1 \cdot (-y_1)} = \frac{1}{1 + y_1}. \tag{20}$$

Example: summary. For the nonlinear system $\dot{x}_1 = x_1^2$, we can take the new variable

$$y_1 = \frac{1}{x_1} - 1$$

for which

$$x_1 = \frac{1}{1 + y_1}.$$

For this new variable, the dynamical system has the form $\dot{y} = -1$.

Does this reduction help solve the original nonlinear system? Not really, since the reduction uses the solution $T_t(x)$ of the system.

But can we have a reduction that does help solve the system? Not really:

- if we know the solution, then, as we have shown, we can easily find the appropriate reduction;
- vice versa, if we know a reduction to a linear system, then, since solutions to linear systems are known, we can easily find the solution to the original nonlinear system.

Because of this, the complexity of finding a reduction to a linear system is exactly the same as the complexity of solving the original nonlinear system.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in

Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395, and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

References

1. R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.
2. K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2021.