Topological Explanation of Why Complex Numbers Are Needed in Quantum Physics

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Abstract In quantum computing, we only use states in which all amplitudes are real numbers. So why do we need complex numbers with non-zero imaginary part in quantum physics in general? In this paper, we provide a simple topological explanation for this need, explanation based on the Second Law of Thermodynamics.

1 Formulation of the Problem

Quantum physics uses general complex numbers. In general, in quantum mechanics, we use complex numbers; see, e.g., [1, 4]. Specifically, if we have classical states $s_1, \ldots, s_n$, then, in quantum physics, we can also have their *superpositions*, i.e., states of the type

$$a_1 \cdot |s_1\rangle + \ldots + a_n \cdot |s_n\rangle, \tag{1}$$

where the coefficients $a_1, \ldots, a_n$ – known as *amplitudes* – are complex numbers for which

$$|a_1|^2 + \ldots + |a_n|^2 = 1. \tag{2}$$

In some physical systems, the amplitudes are real numbers, in others, the corresponding amplitudes have non-zero imaginary parts.

Quantum computing only uses real numbers. One of the exciting applications of quantum physics to computing is quantum computing; see, e.g., [3]. By using

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quantum effects, many efficient algorithms have been invented. Interestingly, all efficient quantum algorithms invented so far only use real-valued amplitudes.

Natural question. The fact that quantum computing only uses real-valued amplitudes prompts the following question: since with only real-valued amplitudes, we can achieve wonders in quantum computing, why does nature need non-real complex values?

What we do in this paper. In this paper, we provide a simple topological explanation for the need for non-real complex numbers. This explanation is based on the general ideas behind the Second Law of Thermodynamics; see, e.g., [1, 4].

2 Second Law of Thermodynamics: A Brief Reminder

General description. According to the Second Law of Thermodynamics, in a closed system, the entropy increases with time – and eventually, the system will reach a stable state, in which there are no more changes.

Clarification. Of course, the above statement needs some clarification:

- First, from the mathematical viewpoint, the system never reaches exactly the stable state, it only asymptotically approaches this state. However, from the physical viewpoint, asymptotical approaching means, in particular, that eventually the actual state becomes so close to the stable state that we cannot distinguish these two states. So, from the observational, physical viewpoint, we *do* reach the final state.
- Second, reaching the stable state may require a large amount of time. For some systems – e.g., for stars – reaching the thermodynamical equilibrium requires billions of year, time period almost as large as the lifetime of the Universe.

3 Resulting Explanation

Let us consider the simplest of non-trivial situations. The more classical states the system has, the more complex its description. Thus, the simplest case is when a system has only one classical state. However, in this case, the physics is trivial: the state does not change in time at all.

The simplest non-trivial case is when the system has two classical state. Without losing generality, we can call one state 0 and the other state 1. In this case, in line with the above general description of quantum states, the possible states of the corresponding quantum system have the form

$$a_0 \cdot |0\rangle + a_1 \cdot |1\rangle, \tag{3}$$

where the complex values a_i must satisfy the condition

$$|a_0|^2 + |a_1|^2 = 1. \tag{4}$$

In general, the amplitudes a_i are complex numbers, i.e., $a_i = b_i + c_i \cdot \mathrm{i}$ for some real numbers b_i and c_i and $\mathrm{i} \stackrel{\text{def}}{=} \sqrt{-1}$. In terms of these real numbers, the condition (4) takes the form

$$b_0^2 + c_0^2 + b_1^2 + c_1^2 = 1. \tag{5}$$

So, in geometric terms, possible states (b_0, c_0, b_1, c_1) form a unit sphere in a 4-D Euclidean space.

Comment. States (3) are a quantum analogue of the states of a 2-state system. In computing, a 2-state system – especially the one where states are called 0 and 1 – is called a *binary digit*, or *bit*, for short. Because of this, in quantum computing, states of type (3) are called *quantum bits*, or *qubits*, for short.

What if we only have real-valued amplitudes. If all amplitudes are real, i.e., if $c_0 = c_1 = 0$, then the condition (5) takes the following simplified form:

$$b_0^2 + b_1^2 = 1. \tag{6}$$

Such states form a unit circle in a plane.

How the state of a qubit changes with time t: simple case. As we have mentioned, in real-valued case, each state of a qubit corresponds to a point in a unit circle. The change in a system corresponds to the point moving along the circle.

One of the possible situations in when the state rotates the central point with the constant speed. In this case, for some values ω and φ, we have

$$a_0 = \cos(\omega \cdot t + \varphi) \text{ and } a_1 = \sin(\omega \cdot t + \varphi). \tag{7}$$

The Second Law of Thermodynamics leads to the desired explanation. According to the Second Law of Thermodynamics, the system (7) – as any other system – should eventually stop changing. In other words, this system should go into some state s that does not change with time.

If we only allow real-valued amplitudes, then all possible states correspond to points on a circle. In particular, the final state s corresponds to some point on a unit circle.

Transition in physical systems are usually continuous. So, we should have a continuous transformation of a circle into itself that would transform the circle into a single point. This clearly is not possible. In topology, this simple fact is described by saying that the homotopy group of the circle is non-trivial; see, e.g., [2].

However, if we allow general complex-valued amplitudes, then such a continuous transition *is* possible: since a circle on a multi-D sphere (already on a 2-D sphere) can be continuously compressed into a point. In topological sense, the homotopy group of the 3-D sphere is trivial.

So, the only way for this simple quantum system to be in accordance with the Second Law of Thermodynamics is to allow complex-valued amplitudes, with non-zero imaginary parts. This explains why such amplitudes are needed in quantum physics – while they seem to be not needed in quantum computing.

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