Algebraic Product Is the Only “And-Like”-Operation for Which Normalized Intersection Is Associative: A Proof

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Abstract For normalized fuzzy sets, intersection is, in general, not normalized. So, if we want to limit ourselves to normalized fuzzy sets, we need to normalize the intersection. It is known that for algebraic product, the normalized intersection is associative, and that for many other “and”-operations (t-norms), normalized intersection is not associative. In this paper, we prove that algebraic product is the only “and”-operation (and even the only “and-like” operation) for which normalized intersection is associative.

1 Formulation of the Problem

Fuzzy sets and normalized fuzzy sets: a brief reminder. A *fuzzy set* on a universal set X is a function $\mu(x)$ that assigns, to each element $x \in X$, a number from the interval $[0,1]$; see, e.g., [1, 3, 4, 5, 6, 7]. Fuzzy sets were invented to describe imprecise (“fuzzy”) natural-language properties such as “small”; the value $\mu(x)$ is then a degree to which, according to the user, the object x has the desired property (e.g., is small).

- The degree 1 means that x definitely has the property.
- The degree 0 means that x definitely does not have this property.
- Intermediate value $\mu(x)$ correspond to x having the property “to some degree”.

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Usually, an additional condition is imposed that the fuzzy set should be *normalized*, i.e., that $\sup\limits_{x \in X} \mu(x) = 1$.

"And"-operations (t-norms): a brief reminder. In many cases, we know the degrees a and b to which the properties A and B are satisfied, and we want to use this information to estimate the degree to which the property "A and B" is satisfied. The function $f_{\&}(a,b)$ that generates the resulting estimate is known as an *"and"-operation*, or, for historical reason, a *t-norm*.

This function have several natural properties. For example, since $A \& B$ means the same as $B \& A$, the corresponding estimates for $A \& B$ and $B \& A$ should coincide, i.e., we should have $f_{\&}(a,b) = f_{\&}(b,a)$ (i.e., in mathematical terms, the "and"-operation should be commutative). Similarly, since $(A \& B) \& C$ means the same as $A \& (B \& C)$, we should always have $f_{\&}(f_{\&}(a,b),c) = f_{\&}(a, f_{\&}(b,c))$ (i.e., in mathematical terms, the "and"-operation should be associative).

Also, intuitively, if A is absolutely true, i.e., if $a = 1$, then our degree of confidence in $A \& B$ is equal to our degree of confidence in B. In precise terms, this means that we should have $f_{\&}(1,b) = b$ for all b. Due to commutativity, we also have $f_{\&}(a,1) = a$ for all a.

Intersection and normalized intersection of fuzzy sets. In general, if we have two properties – as described by the sets S_1 and S_2 of all the elements that satisfy the corresponding property, then the set of all the objects that satisfy the first property *and* satisfy the second property is called the *intersection* $S_1 \cap S_2$ of the two sets. In the fuzzy case, by the meaning of the fuzzy set, the degrees to which an object x satisfies each of the properties are equal to $\mu_1(x)$ and $\mu_2(x)$. Thus, by the meaning of the "and"-operation, the degree to which the first property is satisfied *and* the second property is satisfied is equal to $f_{\&}(\mu_1(x), \mu_2(x))$ for an appropriate "and"-operation $f_{\&}(a,b)$. So, it is reasonable to define the intersection of two fuzzy sets as

$$\mu(x) \stackrel{\text{def}}{=} f_{\&}(\mu_1(x), \mu_2(x)). \tag{1}$$

The problem with this definition is that if we, as usual, limit ourselves to normalized sets, then the above definition may lead to a non-normalized set. For example, if some $a < 1$, on the universal set $X = \{1,2\}$, we consider two functions

$$f_1(1) = a, \;\; f_1(2) = 1, \;\; f_2(1) = 1, \;\; f_2(2) = a,$$

then, in view of the fact that $f_{\&}(a,1) = f_{\&}(1,a) = a$, we get $\mu(1) = \mu(2) = a < 1$, so $\max(\mu(x)) = a < 1$. If want the intersection to be a normalized fuzzy set, we must *normalize* the expression (1), i.e., divide it by the supremum of this expression:

$$(\mu_1 \& \mu_2)(x) = \frac{f_{\&}(\mu_1(x), \mu_2(x))}{\sup\limits_{y \in X} f_{\&}(\mu_1(y), \mu_2(y))}. \tag{2}$$

It is reasonable to call the operation (2) *normalized intersection*.

Comment. The intersection is only defined when the denominator is not equal to 0.

When is normalized intersection associative? By definition (2), normalized intersection operation (2) is commutative. It is known that for algebraic product $f_{\&}(a,b) = a \cdot b$, operation (2) is associative [2]. For many other "and"-operations, normalized intersection is not associative. So, a natural conjecture emerged that the algebraic product is the only "and"-operation for which the normalized intersection is associative.

What we do in this paper. In this paper, we prove that the algebraic product is indeed the only "and"-operation for which the normalized intersection is associative. Moreover, we prove it not only for all "and"-operations, but also for more general binary operations that are not necessarily commutative or associative.

2 Main Result

Definition 1. *By an* "and"-like *operation, we mean a function* $t : [0,1] \times [0,1] \mapsto [0,1]$ *for which* $t(a,1) = t(1,a) = a$ *for all* $a \in [0,1]$.

Comment. This definition is weaker than that of a t-norm. In particular, we do not require monotonicity, or even associativity.

Definition 2. *For each "and"-like operation* $t(a,b)$*, the corresponding* normalized intersection *operation transforms two fuzzy sets* $\mu_1(x)$ *and* $\mu_2(x)$ *into a new fuzzy set*

$$(\mu_1 \,\&\, \mu_2)(x) = \frac{t(\mu_1(x), \mu_2(x))}{\sup\limits_{y \in X} t(\mu_1(y), \mu_2(y))}. \tag{3}$$

Comment. The intersection is only defined when the denominator of the formula (3) is different from 0.

Proposition. *For each "and"-like operation* $t(a,b)$*, the following two conditions are equivalent:*

- $t(a,b)$ *is the algebraic product, i.e.,* $t(a,b) = a \cdot b$*;*
- *The normalized intersection operation corresponding to* $t(a,v)$ *is associative, i.e., for all possible normalized fuzzy sets* μ_1*,* μ_2*, and* μ_3*:*
  - *the fuzzy sets* $(\mu_1 \cap \mu_2) \cap \mu_3$ *and* $\mu_1 \cap (\mu_2 \cap \mu_3)$ *are either both defined or both undefined, and*
  - *if they are both defined, then* $(\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3)$.

Discussion. In this formulation, we only use two properties out of many usual properties of the "and"-operation (t-norm):

- that for every a, we have $t(1,a) = a$, and
- that for every a, we have $t(a,1) = a$.

From the mathematical viewpoint, a natural question is: can we go even further, keep only one of these properties, and still keep our result? The following two simple examples show that both above properties are needed, one is not enough to conclude that $t(a,b) = a \cdot b$:

- The function $t(a,b) = b$ has the property that $t(1,a) = a$ for all a. For this function, as one can easily check, the normalized intersection of μ_1 and μ_2 is simply μ_2: $\mu_1 \cap \mu_2 = \mu_2$, thus $(\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3) = \mu_3$.
- The function $t(a,b) = a$ has the property that $t(a,1) = a$ for all a. For this function, as one can easily check, the normalized intersection of μ_1 and μ_2 is simply μ_1: $\mu_1 \cap \mu_2 = \mu_1$, thus $(\mu_1 \cap \mu_2) \cap \mu_3 = \mu_1 \cap (\mu_2 \cap \mu_3) = \mu_1$.

Proof of the Proposition.

1°. It is known that the normalized intersection operation corresponding to algebraic product is associative. So, to complete the proof, it is sufficient to prove that for each "and"-like operation $t(a,b)$, if the corresponding normalized intersection is associative, then $t(a,b) = a \cdot b$.

2°. Indeed, assume that for an "and"-like operation $t(a,b)$, the corresponding normalized intersection is associative. Let us prove that in this case, $t(a,b) = a \cdot b$.

3°. If $b = 1$, then, by definition of the "and"-like operation, we have $t(a,b) = t(a,1) = a$ and $a \cdot b = a \cdot 1 = a$, so in this case indeed $t(a,b) = a \cdot b$. Thus, to complete the proof, it is sufficient to consider the case when $b < 1$.

4°. For every two numbers $a, b \in [0,1]$ for which $b < 1$, let us consider the following three normalized fuzzy sets on the universal set $X = \{1,2\}$:

$$\mu_1(1) = a, \;\; \mu_1(2) = 1, \;\; \mu_2(1) = 1, \;\; \mu_2(2) = a, \;\; \mu_3(1) = 1, \;\; \mu_3(2) = b.$$

Let us use associativity to prove that $t(a,b) = a \cdot b$.

4.1°. Let us first compute $(\mu_1 \cap \mu_2) \cap \mu_3$. First, for $\mu_1 \cap \mu_2$, by the definition of the "and"-like operation, we get

$$t(\mu_1(1), \mu_2(1)) = t(a,1) = a, \;\; t(\mu_1(2), \mu_2(2)) = t(1,a) = a,$$

thus

$$\max(t(\mu_1(1), \mu_2(1)), t(\mu_1(2), \mu_2(2)) = \max(a,a) = a.$$

In this case, if $a > 0$, then, by the formula (2), we get

$$(\mu_1 \cap \mu_2)(1) = (\mu_1 \cap \mu_2)(2) = \frac{a}{a} = 1.$$

(The case when $a = 0$ is considered in Part 5 of this proof.) Then, for $(\mu_1 \cap \mu_2) \cap \mu_3$, by the same definition of the "and"-like operator, we get

$$t((\mu_1 \cap \mu_2)(1), \mu_3(1)) = t(1,1) = 1, \;\; t((\mu_1 \cap \mu_2)(2), \mu_3(2)) = t(1,b) = b.$$

Thus,

$$\max(t((\mu_1 \cap \mu_2)(1), t((\mu_1 \cap \mu_2)(2)) = \max(1,b) = 1$$

and therefore, by the formula (3), we have

$$((\mu_1 \cap \mu_2) \cap \mu_3)(1) = \frac{1}{1} = 1, \ \ ((\mu_1 \cap \mu_2) \cap \mu_3)(2) = \frac{b}{1} = 1. \tag{4}$$

4.2°. Let us now compute $\mu_1 \cap (\mu_2 \cap \mu_3)$. First, from the definition of the "and"-like operation, we get

$$t(\mu_2(1), \mu_3(1)) = t(1,1) = 1, \ \ t(\mu_2(2), \mu_3(2)) = t(a,b).$$

Here, $t(a,b) \le 1$ – since all the values of the function t are from the interval $[0,1]$. Thus:

$$\max(t(\mu_2(1), \mu_3(1)), t(\mu_2(2), \mu_2(2))) = \max(1, t(a,b)) = 1$$

and therefore, by the formula (2),

$$(\mu_2 \cap \mu_3)(1) = \frac{1}{1} = 1, \ \ (\mu_2 \cap \mu_3)(2) = \frac{t(a,b)}{1} = t(a,b).$$

Now, from the definition of the "and"-like operation, we get

$$t(\mu_1(1), t(\mu_2(1), \mu_3(1))) = t(a,1) = a,$$

$$t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = t(1, t(a,b)) = t(a,b). \tag{5}$$

According to the formula (3), to find the values of the fuzzy set $\mu_1 \cap (\mu_2 \cap \mu_3)$, we need to find the maximum of the two values (5). The value of this maximum depends on which of the two values (5) is larger. Let us consider both possible cases.

4.2.1°. If $a \le t(a,b)$, then

$$\max(t(\mu_1(1), t(\mu_2(1), \mu_3(1))), t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = \max(a, t(a,b)) = t(a,b)$$

and thus, by the formula (3), we get

$$(\mu_1 \cap (\mu_2 \cap \mu_3))(1) = \frac{a}{t(a,b)}, \ \ (\mu_1 \cap (\mu_2 \cap \mu_3))(2) = \frac{t(a,b)}{t(a,b)} = 1.$$

By associativity, these values should be equal to the values (4). By comparing the values of these two fuzzy sets for $x = 2$, we conclude that $b = 1$ which contradicts to our assumption that $b < 1$. Thus, this case is impossible.

4.2.2°. Since, as we have proven, we cannot have $a \le t(a,b)$, we must have $t(a,b) < a$. In this case,

$$\max(t(\mu_1(1), t(\mu_2(1), \mu_3(1))), t(\mu_1(2), t(\mu_2(2), \mu_3(2)))) = \max(a, t(a,b)) = a$$

and thus, by the formula (3), we get

$$(\mu_1 \cap (\mu_2 \cap \mu_3))(1) = \frac{a}{a} = 1, \ \ (\mu_1 \cap (\mu_2 \cap \mu_3))(2) = \frac{t(a,b)}{a}.$$

By associativity, these values should be equal to the values (4). By comparing the values of these two fuzzy sets for $x = 2$, we conclude that

$$\frac{t(a,b)}{a} = b,$$

hence $t(a,b) = a \cdot b$.

For the values $a > 0$, the proposition is proven.

5°. Let us now consider the case when $a = 0$. In this case, by the formulas from Part 4.1 of this proof, the normalized intersection $\mu_1 \cap \mu_2$ is undefined, and, thus, the set $(\mu_1 \cap \mu_2) \cap \mu_3$ is undefined as well. Thus, by associativity, the set $\mu_1 \cap (\mu_2 \cap \mu_3)$ should also be undefined. According to the formulas from Part 4.2 of this proof, we get

$$(\mu_2 \cap \mu_3)(1) = 1, \ \ (\mu_2 \cap \mu_3)(2) = t(0,b).$$

Now, from the definition of the "and"-like operation, we get

$$t(\mu_1(1), t(\mu_2(1), \mu_3(1))) = t(0,1),$$

$$t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = t(1, t(0,b)) = t(0,b). \tag{6}$$

The fact that the fuzzy set $\mu_1 \cap (\mu_2 \cap \mu_3)$ is undefined means that the corresponding denominator is equal to 0:

$$\max(t(\mu_1(1), t(\mu_2(1), \mu_3(1))), t(\mu_1(2), t(\mu_2(2), \mu_3(2))) = \max(t(0,1), t(0,b)) = 0.$$

Thus,

$$0 \le t(0,b) \le \max(t(0,1), t(0,b)) = 0,$$

so indeed $t(0,b) = 0 \cdot b$.

So, the proposition is also proven for the remaining case $a = 0$.

Comment. In our proof, the only assumption that we made about the operation $t(a,b)$ is that $t(a,1) = t(1,a) = a$ for all a. If we additionally assumed that $t(a,b)$ is a t-norm, then the proof would be much shorter; indeed:

- for a t-norm we always have $t(a,b) \le a$, so there is no need to consider the case $t(a,b) > a$, and
- we always have $t(0,b) = 0$, so there is no need to prove this formula.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395 (Center for Collective Impact in Earthquake Science C-CIES), and by the AT&T Fellowship in Information Technology.

It was also supported by the program of the development of the Scientific-Educational Mathematical Center of Volga Federal District No. 075-02-2020-1478, and by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

The authors are thankful to all the participants of the 2023 International Conference on Fuzzy Systems FUZZ-IEEE 2023 (Incheon, Korea, August 13–17, 2023) for valuable discussions.

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