How to Deal with Inconsistent Intervals: Utility-Based Approach Can Overcome the Limitations of the Purely Probability-Based Approach

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Abstract In many application areas, we rely on experts to estimate the numerical values of some quantities. Experts can provide not only the estimates themselves, they can also estimate the accuracies of their estimates – i.e., in effect, they provide an interval of possible values of the quantity of interest. To get a more accurate estimate, it is reasonable to ask several experts – and to take the intersection of the resulting intervals. In some cases, however, experts overestimate the accuracy of their estimates, their intervals are too narrow – so narrow that they are inconsistent: their intersection is empty. In such situations, it is necessary to extend the experts' intervals so that they will become consistent. Which extension should we choose? Since we are dealing with uncertainty, it seems reasonable to apply probability-based approach – well suited for dealing with uncertainty. From the purely mathematical viewpoint, this application is possible – however, as we show, even in simplest situations, it leads to counter-intuitive results. We show that we can make more reasonable recommendations if, instead of only taking into account probabilities, we also take into account our preferences – which, according to decision theory, can be described by utilities.

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1 Formulation of the Problem

Expert knowledge is important. In many application areas, we rely on expert estimates – and economics and finance are a good example of such reliance. We often rely on experts to predict the future value of inflation, the economic growth rate, etc.

Expert estimates often come in interval form. Sometimes expert provide exact numerical values of the corresponding quantities. However, expert estimates are usually very approximate, the estimation errors cannot be ignored.

It is therefore reasonable to make sure that the experts provide us not only with the numerical estimates $\widetilde{x}$ for the quantity of interest x, but also with the estimated upper bound Δ on the estimation error. This means, in effect, that the expert estimates that the actual value x should be somewhere in the interval

$$[\underline{x}, \overline{x}] \stackrel{\text{def}}{=} [\widetilde{x} - \Delta, \widetilde{x} + \Delta].$$

Sometimes, intervals provided by several experts are consistent. Often, we have several experts estimating the same quantity. In many practical situations, intervals $[\underline{x}_i, \overline{x}_i]$ provided by different experts are consistent – i.e., all these intervals have common value. In such situations, if we trust all these experts, it is reasonable to conclude that the actual value belongs to all these intervals – i.e., equivalently, belongs to their intersection.

For example, suppose one reliable witness whose height is 170 cm saw that the suspect is 5 to 15 cm taller than him, i.e., that the suspects's height is in the interval $[175, 185]$. Suppose also that another reliable witness whose height is 185 cm saw that the suspect was 5 to 15 cm shorter then him, i.e., that the suspect's height is somewhere in the interval $[170, 180]$. In this case, it is reasonable to conclude that the actual height of the suspect lies in the intersection $[175, 185] \cap [170, 180] = [175, 180]$ of these two intervals.

Similarly, if one reliable expert predicts that the economy's growth rate will be between 3 and 5 percent, and another one predicts that it will be between 4 and 6 percent, then it is reasonable to conclude that the actual growth rate will belong to the intersection of these two intervals, i.e., to the interval $[3, 5] \cap [4, 6] = [4, 5]$, meaning that the growth rate will be between 4 and 5 percent.

But what if the intervals are inconsistent? In many other cases, the intervals provided by different experts are inconsistent. For example, one expert may be sure that the growth rate will be between 3 and 4 percents, while another expert is sure that it will be between 5 and 6 percents. The corresponding intervals $[3, 4]$ and $[5, 6]$ do not have a common point, so the experts cannot be both right. In other words, at least one of the experts underestimates his/her approximation error. If we enlarge the intervals to take the actual approximation error into account, the intervals will intersect.

The problem: how to extend the intervals. There are many ways to extend the two intervals so that the extended intervals start intersecting. Which one should we choose?

Comment. This problem was explicitly formulated in [2].

What we do in this paper. Our problem is related to uncertainty. So, to solve this problem, it seems reasonable to apply techniques that have been designed to deal with uncertainty and that have been successfully used for several centuries – namely, probability-based techniques. In Section 2, we explain how probability-based approach can be applied to our problem. It turns out, however, that when we apply these techniques to the above problem, we get counter-intuitive results – we show this in Section 3.

The counter-intuitive character of these results is easy to understand: probability-based techniques take care of how frequent different outcomes are, but they do not take into account our preference between different outcomes. To consistently take these preferences into account when making decisions, researchers developed a special *decision theory*, in which preferences are described by numerical values – known as *utilities* assigned to different outcomes. In Section 4, we show if we take utilities into account, then we indeed get a meaningful way to deal with inconsistent intervals. For readers' convenience, all the proofs are placed in a special proofs Section 5.

2 Probability-Based Approach to the Problem

Where do we get probabilities? Strictly speaking, to apply the probability-based approach, we need to have some information about the probabilities. However, in our case, we do not have any such information. The only information that each expert provides is an interval of possible values. We do not know which values from this interval are more probable and which are less probable.

Situations in which we have several alternatives, and we do not know which ones are more probable and which are less probable, are ubiquitous. In such situations, a natural idea is to assign equal probabilities to all these alternatives – i.e., in mathematical terms, to assume that the probability distribution is uniform. This idea makes perfect sense: e.g., if we have three suspects and we have no reasons to consider one of them more probable, then it is reasonable to assign probability 1/3 to each. This idea goes back to the early applications of probabilities and is thus known as *Laplace Indeterminacy Principle*; see, e.g., [5].

In particular, in situations when all we know is that some quantity is located on an interval, a natural idea is to assign equal probability to all the values from this interval – i.e., to assume that the corresponding quantity is uniformly distributed on this interval.

Comment. In our case, we know only that the probability distribution is located on the given interval. Out of all such distributions, we need to select a one that best reflects the given information. Our case is a particular example of a more general situation, when we know a family of possible probability distributions, and out of all these distributions, we need to select a one that best reflects the related uncertainty.

In the interval case, we could select a distribution that is located at one of the points with probability 1 – but by making this selection, we would have eliminated all uncertainty. A more reasonable approach is to select the distribution that preserves the original uncertainty as much as possible. A natural measure of uncertainty is the average number of binary ("yes"-"no") questions that we need to ask to determine the actual value with a given accuracy. It is known (see, e.g., [1, 9]) that this average number is proportional to *entropy* $S \stackrel{\text{def}}{=} -\int f(x) \cdot \ln(f(x))\,dx$ of the probability distribution, where $f(x)$ is the distribution's density. Thus, a natural idea is to select the distribution with largest possible entropy. This idea is known as the *Maximum Entropy* approach; see, e.g., [5]. For the case when all we know is the interval of possible values, the Maximum Entropy approach leads exactly to the uniform distribution on this interval.

Let us formulate our problem in precise terms. We have several intervals $[\underline{x}_i, \overline{x}_i]$ which are inconsistent: $\cap[\underline{x}_i, \overline{x}_i] = \emptyset$. This means that we need to enlarge some of these intervals, i.e., come up with larger intervals $[\underline{X}_i, \overline{X}_i] \supseteq [\underline{x}_i, \overline{x}_i]$ so that the enlarged intervals will have a non-empty intersection $\cap[\underline{X}_i, \overline{X}_i] \neq \emptyset$.

There are many way to perform such an enlargement, which one should we choose? A reasonable idea is to select *the most probable* one. This idea is known as the *Maximum Likelihood* approach; see, e.g., [11]. We assumed that for each interval, the distribution is uniform on this interval. Thus, for each expert, the probability density corresponding to the interval $[\underline{X}_i, \overline{X}_i]$ is equal to

$$\frac{1}{\overline{X}_i - \underline{X}_i}.$$

Different experts are usually independent – if the opinion of an expert is strongly determined by the opinions of others, there is no need to ask this expert anything: his/her opinion is determined from the opinions of others. Since experts are independent, the overall probability of selecting the enlargements is equal to the product of the probabilities $1/(\overline{X}_i - \underline{X}_i)$ corresponding to individual experts. Since we want to select the extension whose probability is the largest, we thus arrive at the following precise formulation.

Probability-based formulation of the problem.

- *Given:* n intervals $[\underline{x}_i, \overline{x}_i]$ whose intersection is empty $\cap[\underline{x}_i, \overline{x}_i] = \emptyset$.
- *Find:* among all *extensions* $[\underline{X}_i, \overline{X}_i] \supseteq [\underline{x}_i, \overline{x}_i]$ for which the intersection is non-empty $\cap[\underline{X}_i, \overline{X}_i] \neq \emptyset$, we need to select an extension with the largest value of the expression
$$\prod_{i=1}^{n} \frac{1}{\overline{X}_i - \underline{X}_i}.$$

3 Limitations of the Probability-Based Approach

Simplest case: description. To show that the above formulation – while seemingly reasonable – leads to counter-intuitive results, let us consider the simplest possible case when we have $n = 2$ non-intersecting intervals. For simplicity, let us assume that both intervals have the same width $w = \overline{x}_i - \underline{x}_i$.

What is reasonable to expect in this case. We have two similar experts, so it is reasonable to expect that they will be treated similarly, i.e., that both expert's intervals will be extended in the same way, i.e., by the same amount. We will show, however, that this is *not* what the probability-based approach recommends.

To show this, let us introduce some notations.

What the probability-based approach recommends for this case.

Proposition 1. *For the case when we have two non-intersecting intervals of equal width w, the probability-based approach means keeping one of the two intervals intact, thus extending only one of the intervals.*

Comment. For readers' convenience, all the proofs are places in a special proof Section 5.

What does this mean? This result means that either the first expert's interval is not extended at all, or the second expert's interval is not extended at all. In both cases, one of the experts is assumed to be absolutely right – which is not what our intuition tells us is reasonable.

4 Utility-Based Approach Helps

Natural idea. Probability-based approach takes into account what is more probable and what is less probable, but it does not take into account what is best for us. To decide which extension is best for us, we need to take into account our preferences.

Decision theory shows (see, e.g., [3, 4, 6, 7, 8, 9, 10]) that preferences of a rational person – i.e., e.g., a person who, if preferring A to B and B to C, also prefers A to C – can be described by assigning, to each possible alternative A a numerical value $u(A)$ in such a way that:

- an alternative A is preferred to an alternative B if and only if $u(A) > u(B)$, and
- for a situation S in which several outcomes $A_1, \ldots, A_n$ are possible, with corresponding probabilities $p_1, \ldots, p_n$, the utility $u(S)$ is equal to

$$u(S) = p_1 \cdot u(A_1) + \ldots + p_n \cdot u(A_n).$$

Utility can be defined as follows:

- we select a very bad alternative A_- that is worse that anything that we may actually encounter, and

- we select a very good alternative A_+ that is better than anything that we may actually encounter.

Then, for each alternative A, we ask the user to compare this alternative with a lottery $L(p)$ in which the user:

- will get A_+ with probability p, and
- will get A_- with remaining probability $1 - p$.

Here:

- For small probabilities p, we have $L(p) \approx A_-$ and thus, due to our choice of A_-, the lottery $L(p)$ is worse than A; we will denote it by $L(p) < A$.
- For probabilities p close to 1, we have $L(p) \approx A_+$ and thus, due to our choice of A_+, the lottery A is worse than $L(p)$; $A < L(p)$.

Clearly, if $p < p'$, then $L(p) < L(p')$. So:

- if $p < p'$ and $L(p') < A$, then $L(p) < A$, and
- if $p < p'$ and $A < L(p)$, then $A < L(p')$.

Thus, as one can show, there exists a threshold value u such that:

- for $p < u$, we have $L(p) < A$, and
- for $p > u$, we have $A < L(p)$.

This threshold value u is exactly the utility $u(A)$ of the alternative A.

The numerical value $u(A)$ of the utility depends on the selection of the two alternatives A_- and A_+. It turns out that if we select a different pair of alternatives, then the new utility function $U(A)$ is related to the original one $u(A)$ by a linear expression: $U(A) = a_0 + a_1 \cdot u(A)$ for some constants a_0 and $a_1 > 0$.

Let us apply this idea to our case. In general, the narrower the interval, the more information it carries about the quantity. So, for a single interval, utility u depends on the width w of the interval: the larger the width w, the smaller the utility $u(w)$.

As we have mentioned, experts are independent. For independent events, as shown, e.g., in [3, 4], utility is equal to the sum of the utilities. Thus, it makes sense to describe the utility of an extension as the sum of the utilities $u(W_i)$ corresponding to intervals $[\underline{X}_i, \overline{X}_i]$ with widths $W_i = \overline{X}_i - \underline{X}_i$.

By definition of utility, the best alternative is the one for which utility is the largest. Thus, we arrive at the following precise formulation of the problem:

Preliminary utility-based formulation of the problem

- *Given:* n intervals $[\underline{x}_i, \overline{x}_i]$ whose intersection is empty $\cap[\underline{x}_i, \overline{x}_i] = \emptyset$.
- *Find:* among all *extensions* $[\underline{X}_i, \overline{X}_i] \supseteq [\underline{x}_i, \overline{x}_i]$ for which the intersection is non-empty $\cap[\underline{X}_i, \overline{X}_i] \neq \emptyset$, we need to select an extension with the largest value of the expression

$$\sum_{i=1}^{n} u(\overline{X}_i - \underline{X}_i).$$

What are reasonable utility functions: mathematical description of the problem. To use the above criterion, we need to find out what are the reasonable utility functions $u(w)$. To find this out, let us take into account that we are talking expert estimates of different quantities such as height. For most of these quantities, the numerical value depends of the choice of a measuring unit. For example, if we replace centimeters with meters, then all numerical values are divided by 100: 175 cm becomes 1.75 m. Usually, there is no preferable measuring unit; which measuring unit to choose is a matter of convenience. In this case, it is reasonable to require that our preferences do not depend on this choice.

In mathematical terms, if we replace the original measuring unit with a new unit which is λ times smaller, then all numerical values – including numerical values of each width w – are multiplied by λ: $w \mapsto \lambda \cdot w$. So, if in the original units, we had width w, the same width in the new units is described as $\lambda \cdot w$. So, the same preferences that in the original units were described by the utility function $u(w)$ are now described by a new function $u(\lambda \cdot w)$.

These two utility functions should describe the preference relation. As we have mentioned, this means that they should be linearly related. In other words, for every $\lambda > 0$, there should be some values $a_0(\lambda)$ and $a_1(\lambda)$ for which, for every $w > 0$, we have

$$u(\lambda \cdot w) = a_0(\lambda) + a_1(\lambda) \cdot u(w). \quad (1)$$

What are reasonable utility functions: from mathematical description to explicit formulas. It is reasonable to assume that small changes of width lead to equally small changes in the utility, i.e., that the dependence $u(w)$ is smooth (differentiable).

Definition 1. *We say that two utility functions $u(w)$ and $U(w)$ are* equivalent *if there exist constant a_0 and $a_1 > 0$ for which, for all w, we have*

$$U(w) = a_0 + a_1 \cdot u(w).$$

Proposition 2. *Every decreasing differentiable function $u(w)$ that satisfies the equation (1) for all $w > 0$ and for all $\lambda > 0$ is equivalent either to $-\ln(w)$ or to $-\text{sign}(b) \cdot w^b$ for some $b \neq 0$.*

Proposition 3. *For $u(w) = -\ln(w)$, the utility-based formulation is equivalent to the probability-based formulation.*

Since, as we have shown earlier, the probability-based formulation leads to counter-intuitive results, the case when $u(w)$ is equivalent to $-\ln(w)$ is also counter-intuitive. Hence, we should only consider the power-law utility functions. Thus, the above formulation takes the following form:

Final utility-based formulation of the problem.

- *Given:* n intervals $[\underline{x}_i, \overline{x}_i]$ whose intersection is empty $\cap[\underline{x}_i, \overline{x}_i] = \emptyset$.

- *Find:* among all *extensions* $[\underline{X}_i, \overline{X}_i] \supseteq [\underline{x}_i, \overline{x}_i]$ for which the intersection is non-empty $\cap[\underline{X}_i, \overline{X}_i] \neq \emptyset$, we need to select an extension with the largest value of the expression
$$-\text{sign}(b) \cdot \sum_{i=1}^{n} (\overline{X}_i - \underline{X}_i)^b.$$

The resulting utility-based description is in perfect accordance with our intuition. Indeed, the following result holds:

Proposition 4. *For the case when $b > 1$ and when we have two non-intersecting intervals of equal width w, the utility-based approach means that we extend both intervals by the same additional width.*

Comment. In this paper, we focused on *direct* inconsistency, when several expert estimates of the same quantity are inconsistent. The same approach can be applied to the case of *indirect* inconsistency, when we know the relation between several quantities, and expert estimates of these quantities are inconsistent with this relation.

For example, values of current I, voltage V, and resistance R must satisfy Ohm's law $V = I \cdot R$. In this case, interval estimates $[0.9, 1.1]$ for I and R and $[0.7, 0.8]$ for V are inconsistent – since in this case, possible values of the product $I \cdot R$ range from $0.9 \cdot 0.9 = 0.81$ to $1.1 \cdot 1.1 = 1.21$ and thus cannot be smaller than or equal to 0.8.

In such situations, we can use the same approach to select the best extension of intervals, the only difference from above formulation is that the consistency condition will be more complicated.

5 Proofs

Proof of Proposition 1. Without losing generality, we can assume that the first interval is to the left of the second one, i.e., that $\overline{x}_1 < \underline{x}_2$. Let us denote the distance between the given intervals by $d \stackrel{\text{def}}{=} \underline{x}_2 - \overline{x}_1$.

Maximizing the above product is equivalent to minimizing the product of the widths $\overline{X}_i - \underline{X}_i$ of the enlarged intervals.

Since the interval $[\underline{X}_1, \overline{X}_1]$ extends the first given interval, we must have $\underline{X}_1 \leq \underline{x}_1$. If we have $\underline{X}_1 < \underline{x}_1$, then we can replace $\underline{X}_1$ with $\underline{x}_1$ and thus, get intersecting extensions with smaller width of the first interval – and hence, smaller product of widths. Thus, the smallest product of widths is attained when $\underline{X}_1 = \underline{x}_1$.

Similarly, we can conclude that the smallest product of widths is attained when $\overline{X}_2 = \overline{x}_1$. In this case, the fact that the intervals intersects means that $\overline{X}_1 \geq \underline{X}_2$.

If we have $\overline{X}_1 > \underline{X}_2$, then we can shorten the first extended interval to the upper endpoint $\overline{X}_1 = \underline{X}_2$, while keeping the two extended intervals intersecting. After this shortening, the width of the first extended interval decreases while the width of the second one remains the same – thus, the product of the widths decreases. Therefore, in situations when the product of widths is the smallest possible, we cannot have $\overline{X}_1 > \underline{X}_2$ – thus, we must have $\overline{X}_1 = \underline{X}_2$.

So, in the most probable case, the two intervals divide the distance d between themselves: the first interval has width $w + d_1$, where we denote $d_1 \stackrel{\text{def}}{=} \overline{X}_1 - \overline{x}_1$, and the second interval has width $w + d - d_1$. The smallest product of the widths corresponding to the smallest value of the product $(w + d_1) \cdot (w + d - d_1)$. One can easily check that this quadratic function attains its maximum for $d_1 = d/2$, and that this expression is increasing for $d_1 < d/2$ and decreasing for $d_1 > d/2$. Thus, for values $d_1 \in [0, 1]$, the product attains the smallest value if either when $d_1 = 0$ or when $d_1 = d$.

What does this mean? The case $d_1 = 0$ means that the first expert's interval is not extended at all, the case $d_1 = d$ means that the second expert's interval is not extended at all. The proposition is thus proven.

Proof of Proposition 2. Let us first show that the functions $a_0(\lambda)$ and $a_1(\lambda)$ are also differentiable. Indeed, let us consider the formula (2) for two values w_1 and w_2:

$$u(\lambda \cdot w_1) = a_0(\lambda) + a_1(\lambda) \cdot u(w_1), \tag{2}$$

$$u(\lambda \cdot w_2) = a_0(\lambda) + a_1(\lambda) \cdot u(w_2). \tag{3}$$

Subtracting (3) from (2), we get

$$u(\lambda \cdot w_2) - u(\lambda \cdot w_1) = a_1(\lambda) \cdot (u(w_2) - u(w_1)),$$

hence

$$a_1(\lambda) = \frac{u(\lambda \cdot w_2) - u(\lambda \cdot w_1)}{u(w_2) - u(w_1)}. \tag{4}$$

Since the function $u(w)$ is differentiable, the right-hand side of the formula (4) is differentiable as well and thus, its left-hand side – i.e., the function $a_1(\lambda)$ – is also differentiable.

From the formula (2), we can now conclude that the function

$$a_0(\lambda) = u(\lambda \cdot w_1) - a_0(\lambda) + a_1(\lambda) \cdot u(w_1)$$

is the difference of two differentiable functions and is, thus, differentiable as well.

Since all three functions used in formula (1) are differentiable, we can differentiate both sides with respect to λ. This will lead us to the following expression:

$$w \cdot u'(\lambda \cdot w) = a_0'(\lambda) + a_1'(\lambda) \cdot u(w).$$

In particular, for $\lambda = 1$, we get:

$$w \cdot u'(w) = c_0 + c_1 \cdot u(w), \tag{5}$$

where we denoted $c_i \stackrel{\text{def}}{=} a_i'(1)$.

If both c_0 and c_1 were equal to 0, we would get $u'(w) = 0$ and thus, $u(w)$ would be a constant, but we assumed that the function $u(w)$ is a decreasing function of the width w. Thus, the expression $c_0 + c_1 \cdot u(w)$ is not identically 0.

The formula (5) can be reformulated as

$$w \cdot \frac{du}{dw} = c_0 + c_1 \cdot u.$$

We can separate the variables in this equality if we:

- multiply both sides by dw,
- divide both sides by w, and
- divide both sides by $c_0 + c_1 \cdot u$.

Then, we get the following equality:

$$\frac{du}{c_0 + c_1 \cdot u} = \frac{dw}{w}. \tag{6}$$

If $c_1 = 0$, then integrating both sides of the formula (6) leads to

$$\frac{u}{c_0} = \ln(w) + C,$$

where C denotes the integration constant, thus in this case $u(w) = c_0 \cdot \ln(w) + c_0 \cdot C$. Since the function $u(w)$ should be decreasing, we must have $c_0 < 0$, so this function is equivalent to $-\ln(w)$.

When $c_1 \neq 0$, we have $d(c_0 + c_1 \cdot u) = c_1 \cdot du$, so

$$du = \frac{d(c_0 + c_1 \cdot u)}{c_1}. \tag{7}$$

Substituting the expression (7) into the formula (6), we get

$$\frac{1}{c_1} \cdot \frac{d(c_0 + c_1 \cdot u)}{c_0 + c_1 \cdot u} = \frac{dw}{w}.$$

Integrating both sides, we get

$$\frac{1}{c_1} \cdot \ln(c_0 + c_1 \cdot u) = \ln(w) + C,$$

hence

$$\ln(c_0 + c_1 \cdot u) = c_1 \cdot \ln(w) + c_1 \cdot C.$$

Applying $\exp(x)$ to both sides, we get

$$c_0 + c_1 \cdot u(w) = \exp(c_1 \cdot C) \cdot \cdot w^{c_1},$$

i.e., that

$$u(w) = \frac{\exp(c_1 \cdot C)}{c_1} \cdot w^{c_1} - \frac{c_0}{c_1}.$$

Thus, the utility function $u(w)$ is equivalent to the power law $u(w) = \pm w^b$ for some value b. This function should be decreasing, so:

- for $b > 0$, we take the minus sign, and
- for $b < 0$, we take the plus sign.

In general, $u(w) = -\text{sign}(b) \cdot w^b$.

Proof of Proposition 3. For the function $u(w) = -\ln(w)$, maximizing the expression (1) is equivalent to maximizing the expression $E \stackrel{\text{def}}{=} -\sum\limits_{i=1}^{n} \ln(W_i)$. Since the function $\exp(x)$ is increasing, this is equivalent to maximizing $\exp(E)$. One can show that $\exp(E)$ is exactly what we maximized in the maximum likelihood approach. The proposition is proven.

Proof of Proposition 4. Similarly to the proof of Proposition 1, we can show that in this case, the utility solution leads to $W_1 = w + d_1$ and $W_2 = w + d - d_1$. In this case, the utility formulation requires us to maximize the expression

$$-(w + d_1)^b - (w + d - d_1)^b.$$

According to calculus, to find the largest value of this expression on the interval $d_1 \in [0, d]$, we need to compare the values of this expression at the endpoints of this interval, i.e., for $d_1 = 0$ and $d_1 = d$, and at a point d_1 at which the derivative of this expression is equal to 0. Differentiating this expression with respect to d_1 and equating the derivative to 0, we get $d_1 = d/2$. For $b > 1$, the value corresponding to $d_1 = d/2$ is larger than the values corresponding to $d_1 = 0$ and $d_1 = d$ – due to convexity of the function x^b. Thus, the maximum utility is indeed attained when both intervals are extended equally.

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