Why Micro-Funding? Why Small Businesses Are Important? Analysis Based on First Principles

Hien D. Tran, Edwin Tomy George, and Vladik Kreinovich

Abstract On the one hand, in economics, there is a well-known and well-studied economy of scale: when two smaller companies merge, it lowers their costs and thus, makes them more effective and therefore more competitive. At first glance, this advantage of big size would make economy dominated by big companies – but in reality, small business remain a significant and important economic sector. Similarly, it is well known and well studied that research collaboration enhances researchers' productivity – but still a significant portion of important results come from individual efforts. In several applications areas, there are area-specific explanations for this seemingly contradictory phenomenon. In this paper, we provide a general explanation based on first principles. Our reasoning also leads to a new explanation of the ubiquity of Zipf's Law – a wa that describes, e.g., the distribution of companies by size.

1 Formulation of the problem

Economy of scale: well-known and well-studied phenomenon. In economics, there is a known phenomenon of *economy of scale*, when a merger of two small companies helps lower the costs.

The same phenomenon is known in all kinds of activities. For example, when researchers collaborate, they can usually achieve much more than when they work on their own or in small groups.

Based on this logic, one would expect that this effectiveness leads to the dominance of big companies in economics and big well-funded projects in science.

Hien D. Tran
Tan-Tao University, Ho Chi Minh City, Vietnam, e-mail: hien.tran@ttu.edu.vn

Edwin Tomy George and Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso
500 W. University, El Paso, Texas 79968, USA
e-mail: etomygeorg@miners.utep.edu, vladik@utep.edu

However, many small businesses are successful. In practice, however, there is a stable and significant proportion of small businesses – which shows that there is economic benefit in having small businesses in addition to big companies.

Micro-funding is useful.

- Along the same lines, it has been empirically shown that the best way to stimulate economy is to provide funding both to big and small businesses, i.e., to combine macro-funding and micro-funding.
- Similarly, when supporting science, the best effect is achieved when usual-size grants are supplemented by micro-funding, i.e., by smaller-size grants; see, e.g., [2, 3, 7] and references therein.

How can we explain this phenomenon? In economics, in science sponsorship, and in other similar areas there are good explanations for this phenomenon.

Remaining problem. However, the current explanations are specific to each area, while the phenomenon is the same in all these areas. It is therefore desirable to look for a general explanation for this phenomenon.

What we do in this paper. In this paper, we provide such a general explanation.

Comment. Preliminary results from this paper first appeared in [8].

2 Analysis of the Problem and the Resulting Explanation

Let us formulate the problem in precise terms. In all such situations:

- we have a fixed amount of money, and
- we want to find the best way to distribute this amount.

Each distribution can be naturally described by a density function $f(m)$ for which the number of grants of sizes between m and Δm is equal to $f(m)\cdot\Delta m$. In these terms, the question is: What is the optimal function $f(m)$?

What do we mean by optimal: towards a precise definition. In the above formulation, we used the word "optimal". Usually, this means that we have an objective function – e.g.:

- profit for a company,
- research productivity for a group of researchers,

and we want to maximize the value of this function. However, in our case, we do not know the exact form of the objective function.

All we know is that:

- some distributions are more effective than others; we will denote this relation by $f(m) \succ g(m)$ – and

- some distributions are of the same effectiveness as others; we will denote this relation by $f(m) \sim g(m)$.

These relations must be consistent. If f is better than g and g is better than h, then we should be able to conclude that f is better than h. This leads to the following definition.

Definition 1. *By an* optimality criterion *on a set A, we mean a pair of binary relations* $\langle \succ, \sim \rangle$ *that satisfy the following properties for all* $a, b, c \in A$*:*

- *if* $a \succ b$ *and* $b \succ c$*, then* $a \succ c$*;*
- *if* $a \succ b$ *and* $b \sim c$*, then* $a \succ c$*;*
- *if* $a \sim b$ *and* $b \succ c$*, then* $a \succ c$*;*
- *if* $a \sim b$ *and* $b \sim c$*, then* $a \sim c$*;*
- *if* $a \sim b$*, then* $b \sim a$*;*
- $a \sim a$*; and*
- *if* $a \succ b$ *then we cannot have* $a \sim b$*.*

In these terms, an optimal alternative is an alternative which is better than – or at least of the same quality – as all other alternatives.

Definition 2. *We say that an element* $a_0 \in A$ *is* optimal *with respect to the optimality criterion* $\langle \succ, \sim \rangle$ *if for every* $a \in A$*, we have either* $a_0 \succ a$ *or* $a_0 \sim a$*.*

Optimality criterion must be final. It is reasonable to require that there is one and only one optimal alternative. Indeed, if there are no optimal alternatives, then this criterion is useless. So, there must be at least one optimal alternative.

On the other hand, if there are two or more alternatives of equal quality, then we can use this non-uniqueness to optimize something else. For example, if a company has two alternatives with the same amount of expected profit, it is reasonable to select one with the lowest risk, etc. This means, in effect, that in this case our original optimality criterion $\succ$ was not final, and we add additional criterion $\succ_a$ to formulate a new criterion $\succ_n$ for which $a \succ_n b$ if and only if:

- either $a \succ b$ according to the original criterion,
- or $a \sim b$ according to the original criterion and $a \succ_a b$ according to the additional criterion.

If the new criterion still leads to several equally good optimal alternatives, this means that this new criterion is still not final: we can use this non-uniqueness to optimize something else. This process can continue until we reach the final optimality criterion, i.e., a criterion for which there is exactly one optimal alternative. This leads to the following definition.

Definition 3. *We say that the optimality criterion is* final *if there exists exactly one alternative that is optimal with respect to this criterion.*

Optimality criterion should be scale-invariant. Previously, we talked about general properties of optimality criteria. Let us now consider our case, when alternatives are non-negative functions $f(m)$ defined for positive values m.

Each value m describes the amount of funding. The numerical value of funding depends on what units we choose for counting money: we can use dollars, we can use Euros, we can use Mexican pesos, etc. The choice of a monetary unit is a matter of convenience. It is therefore reasonable to require that the relative quality of different functions should not depend on what unit we use for counting.

By definition of density, in the original units, for each pair of values m and $\Delta m \ll m$, we have $f(m) \cdot \Delta m$ folks who receive grants with amounts from m to $m + \Delta m$. If we replace one monetary unit with another monetary unit that is λ times smaller, then all the numerical values will be multiplied by λ: $m \mapsto m' = \lambda \cdot m$ and $\Delta m \mapsto \Delta m' = \lambda \cdot \Delta m$. Let us denote by $f'(m')$ the probability density as described in the new units. In the new units, the same number of people $f(m) \cdot \Delta m$ is now equal to $f'(m') \cdot \Delta m'$. So,

$$f'(m') \cdot \Delta m' = f(m) \cdot \Delta m. \tag{1}$$

Here, $m = m'/\lambda$ and $\Delta m = \Delta m'/\lambda$. Substituting these expressions for m and Δm into the formula (1), we conclude that

$$f'(m') \cdot \Delta m' = f(m'/\lambda) \cdot \Delta m'/\lambda.$$

Dividing both sides by $\Delta m'$, we conclude that the density in the new units has the form $f'(m') = f(m'/\lambda)/\lambda$.

Thus, the dependence described, in the original unit, by a function $f(m)$ will now be described by a function $S_\lambda(f)$ for which $(S_\lambda(f))(m) \stackrel{\text{def}}{=} f(m/\lambda)/\lambda$. So, the requirement that the relative quality not depend on the unit takes the following form:

Definition 4. *Let A be the set of all non-negative functions $f(m)$ defined for all positive values m. We say that an optimality criterion defined on this set is* scale-invariant *if for every two function $f, g \in S$, and for every $\lambda > 0$, the following two conditions hold:*

- *if $f \succ g$, then $S_\lambda(f) \succ S_\lambda(g)$, and*
- *if $f \sim g$, then $S_\lambda(f) \sim S_\lambda(g)$.*

Now, we are ready to formulate our main result.

Proposition. *For every scale-invariant final optimality criterion, the optimal function has the form $f(m) = C/m$ for some value C.*

Comment. For readers' convenience, the proof of this result is placed in a special Proof section.

Of course, our model is a simplification. Of course, as usual in numerical analysis of real-life problems, our description is an idealization: we assume that the monetary amount can take any real values, but values which are too small or too large are not realistic:

- we cannot give a loan of 0.1 cents, and,

- for a different reason, we cannot give a loan of a thousand trillion dollars – no one has that much money.

However, similar simplifications work well in many other applications, so we believe that our result well describes what is happening for realistic value m.

This leads us to the following conclusion.

Main conclusion. The optimal function – as described by the above proposition – is everywhere positive. So in the optimal arrangement, we should always have some grants with small m.

This explains the ubiquity and effectiveness of micro-funding.

This leads to a new explanation for the ubiquity of Zipf's Law. The specific form of the optimal function $f(m)$ is known as Zipf's Law; see, e.g., [5] and references therein. The law is ubiquitous: in particular, it describes the distribution of the companies by size – one of the phenomena that we are trying to explain. Thus, our result provide one more explanation of Zipf's law; for other explanations, see, e.g., [1, 4].

3 Proof

We assumed that the optimality criterion is final. This means that there exists exactly one function that is optimal with respect to this criterion. Let us denote this optimal function by F.

Let us first prove that the function F is itself scale-invariant, i.e., that for every $\lambda > 0$, we have $F = S_\lambda(F)$. Indeed, by definition of optimality, the fact that F is optimal means that for every $a \in A$ we have either $F \succ a$ or $F \sim a$. In particular, for every function a, we have $F \succ S_{1/\lambda}(a)$ or $F \sim S_{1/\lambda}(a)$. By scale-invariance, we can conclude that $S_\lambda(F) \succ S_\lambda(S_{1/\lambda}(a))$ or $S_\lambda(F) \sim S_\lambda(S_{1/\lambda}(a))$.

One can easily see that $S_\lambda(S_{1/\lambda}(a)) = a$. Thus, for every $a \in A$, we have either $S_\lambda(F) \succ a$ or $S_\lambda(F) \sim a$. By definition of optimality, this means that the function $S_\lambda(F)$ is optimal. However, we assumed that the optimality criterion is final, which means that there is only one optimal function. Thus indeed $S_\lambda(F) = F$ for all $\lambda > 0$.

By definition of the expression $S_\lambda(F)$, thus means that for every $m > 0$ and every $\lambda > 0$, we have $F(m/\lambda)/\lambda = F(m)$. In particular, for $\lambda = m$, we get $F(m) = C/m$, where we denoted $C \stackrel{\text{def}}{=} F(1)$. The proposition is proven.

Comment. The main ideas behind this proof were first used in [1] and [6].

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