Giant Footprints of Buddha and Generalized Limits

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Abstract In many places in Asia, there are footprints claimed to be left by Buddha. Many of them are much larger than the usual size of human feet, up to 150 cm and more in length. In this paper, we provide a possible mathematical explanation for such unusual sizes.

1 Formulation of the Problem

In many places in Asia, there are footprints claimed to be left by Buddha. Many of them are much larger than the usual size of human feet, up to 150 cm and more in length; see, e.g., [1, 2, 3, 4, 5, 6]. How can we explain such unusual sizes?

2 Our Idea and the Resulting Explanation

Specific feature of supernatural beings. One of the features of supernatural beings is that they can travel to infinity and beyond, not bounded by usual limitations on velocity, they can travel to infinity and beyond in finite time. How can we describe the effect of such a travel? How does such a travel affect corresponding quantities?

Suppose that we know the values $v_0, v_1, v_2, \ldots, v_n, \ldots$ of some quantity v at spatial locations at distances equal to 0, 1, 2, ..., n, ... units from the starting point: e.g., the mass of an object, the linear size of an object, etc. If this sequence tends to the

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limit $v_{\rm lim} = \lim v_n$, then it is reasonable to say that the value v_∞ of the quantity v at the infinity point is equal to this limit.

But what if for a sequence v_n, there is no limit? To answer this question, we need to consider what is usually called *generalized limits*, i.e., extend the limit operation to other sequences. For simplicity, let us consider bounded sequences, i.e., sequences v_n for which, for some bound B, we have $|v_n| \le B$ for all n.

Definition 1.

- *We say that a sequence of real numbers $v_0, v_1, \ldots, v_n, \ldots$ is called* bounded *if there exists a real number B for which $|v_n| \le B$ for all n.*
- *By a* generalized limit $g(v)$*, we mean a mapping that assigns, to each bounded sequence $v = \{v_n\}$, a real number such that if the sequence x_n tends to a limit $v_{\rm lim}$, then $g(v) = v_{\rm lim}$.*

What are the reasonable properties of such a generalized limit?

x-shift-invariance. The first reasonable property comes from the fact that the generalized limit should not change if we simply consider a different starting point. Instead of starting with the original point, we can start with the point at 1 unit distance, the point at which the value of the corresponding quantity is equal to v_1. If we take this point as the starting point – and if the motion is along the straight line – then with respect to this new starting point:

- the new value v'_0 is equal to the previous value v_1,
- the new value v'_1 is equal to the previous value v_2, and,
- in general, the new value v'_n is equal to the original value v_{n+1}.

The natural requirement is that the generalized limit should not change if we consider the spatially-(x-)shifted sequence v' instead of the original sequence v.

Definition 2. *We say that the generalized limit is x-*shift-invariant *if for every sequence v, we have $g(v) = g(v')$, where $v'_n \stackrel{\rm def}{=} v_{n+1}$.*

Additional properties. We are discussing how the numerical values of physical quantities can change under such a supernatural transition. In this analysis, it is important to take into account that for physical quantities, the numerical values depends on the choice of the starting point and on the choice of the measuring unit.

- If we replace the original starting point with a new one which is v_0 values earlier – e.g., if we go from Celsius to Kelvin temperature scale – then the value x_0 is added to all the numerical values: instead of the original value x, we get a new value $x' = x + x_0$.
- Similarly, if we replace the original measuring unit with the one that is a times smaller – e.g., if we replace meters with centimeters – then all the numerical value are multiplied by a: instead of the original value x, we get a new value $x' = a \cdot x$.

It seems reasonable to require that the limit process should not change if we simply change the starting point or the measuring unit. In both cases, we can make the same change for all the moments of time – or, alternatively, we can apply somewhat different changes at different moments of time.

- For *shift* $x \mapsto x + x_0$, this means that if we start with a sequence $\{v_n\}$ and we apply shifts $\{v'_n\}$, then the limit value $g(v + v')$ of the shifted sequence $\{v_n + v'_n\}$ should be obtained from the limit value $g(v)$ of the original sequence by adding the limit shift $g(v')$: $g(v + v') = g(v) + g(v')$.
- For *scaling* $x \mapsto a \cdot x$, this means that if we start with a sequence $\{v_n\}$ and we apply scalings $\{v'_n\}$, then the limit value $g(v' \cdot v)$ of the re-scaled sequence $\{v'_n \cdot v_n\}$ should be obtained from the limit value $g(v)$ of the original sequence by multiplying this limit value by the limit value $g(v')$ of scaling: $g(v' \cdot v) = g(v') \cdot g(v)$.

Thus, we arrive at the following definitions.

Definition 3. *We say that the generalized limit is* v-shift-invariant *if for every two sequences v and v', we have $g(v + v') = g(v) + g(v')$.*

Definition 4. *We say that the generalized limit is* v-scale-invariant *if for every two sequences v and v', we have $g(v' \cdot v) = g(v') \cdot g(v)$.*

Proposition.

- *There exists an x-shift-invariant and v-shift-invariant generalized limit.*
- *It is not possible to have an x-shift-invariant and v-scale-invariant generalized limit.*

Discussion. This result shows that supernatural travel to infinity and beyond cannot be v-scale-invariant, i.e., in general, it does not preserve the sizes: these sizes may increase or decrease. This provides a possible explanation for the fact that Buddha – who was normal size before possible supernatural trips – could leave giant footsteps upon his later visits.

Proof.

1°. Let us first prove that there exists an x-shift-invariant and v-shift-invariant generalized limit. We will actually prove a slightly stronger statement, that it is possible to have a generalized limit that, in addition to these two invariance properties, is also *homogeneous* in the sense that $g(c \cdot v) = c \cdot g(v)$ for all real numbers c, where $c \cdot \{v_n\} \stackrel{\text{def}}{=} \{c \cdot v_n\}$.

Indeed, one can easily check that the set $\mathcal{S}$ of all bounded sequences forms a linear space. In these terms, v-shift-invariance and homogeneity mean that we want to construct is a linear function from this space to real numbers.

The x-shift-invariance means that we should have $g(v) = g(v')$ for $v' = \{v_{n+1}\}_n$. Since the desired function $g(v)$ is linear, this is equivalent to requiring that we should have $g(d) = 0$, where we denoted $d \stackrel{\text{def}}{=} v' - v$. For each bounded sequence v with bound B, the sequence

$$d = \{v_2 - v_1, v_3 - v_2, \ldots, v_{n+1} - v_n, \ldots\}$$

has the property that for each n, the sum of the first n terms in this sequence is bounded by $2B$. Indeed,

$$d_1 + d_2 + \ldots + d_n = (v_2 - v_1) + (v_3 - v_2) + \ldots + (b_{n+1} - v_n) = v_{n+1} - v_1.$$

So, since $|v_{n+1}| \le B$ and $|v_1| \le B$, we have

$$|d_1 + \ldots + d_n| = |v_{n+1} - v_1| \le 2B.$$

Vice versa, if we have a sequence $\{d_1, d_2, \ldots\}$ for which the sums of the first n terms in this sequence are bounded by the same constant B, then this sequence can be represented as $d_n = v_{n+1} - v_n$, where

$$v_n = d_1 + \ldots + d_{n-1}$$

is the sequence bounded by the same value B.

Thus, x-shift-invariance means that we should have $g(d) = 0$ for all the sequences d for which the sums of the first n terms in this sequence are bounded by the same constant. One can easily check that such sequences also form a linear space; let us denote it by $\mathcal{B}$. So, in linear algebra terms, what we want is a linear function on the factor-space $\mathcal{S}/\mathcal{B}$. The desired function should extend the usual limit.

We can form a basis in the set of all the sequences that have a limit; for this basis, the value $g(v)$ is determined – it is equal to the limit of the sequence v. We can extend this basis to the whole factor-space, and assign arbitrary values $g(e)$ to all the new elements of this basis – e.g., we can take $g(e) = 0$ for all these elements. Then, every element V of the factor-space can be represented as $V = \sum c_i \cdot e_i$ for a finite number of real numbers c_i and basis elements e_i. It is easy to see that the function g that assigns, to each such element, the value $g(V) \stackrel{\text{def}}{=} \sum c_i \cdot g(e_i)$ is the desired x-shift-invariance, v-shift-invariant, and homogeneous function.

2°. Let us now prove that a x-shift-invariant generalized limit cannot be v-scale-invariant. We will prove this by contradiction. Let us assume that the generalized limit $g(v)$ is both x-shift-invariant and v-scale-invariant. Then, for the sequence $v = \{1, -1, 1, -1, \ldots\}$ of alternating values 1 and -1, the x-shift leads to $v' = \{-1, 1, -1, \ldots\}$. By x-shift invariance, we have $g(v) = g(v')$.

One can easily see that $v' = v(-1) \cdot v$, where $v(-1)$ denotes a sequence $\{-1, -1, \ldots\}$ all elements of which are equal to -1. The sequence $v(-1)$ clearly has a limit -1, so, by definition of the generalized limit, we have $g(v(-1)) = -1$. By v-scale-invariance, we get $g(v') = g(v(-1)) \cdot g(v)$, i.e., $g(v') = -g(v)$. We already know that $g(v) = g(v')$, so $g(v) = -g(v)$ and thus, $g(v) = 0$.

Now, if we multiply the sequence v by itself, we get a sequence $v(1) \stackrel{\text{def}}{=} \{1, 1, \ldots\}$ consisting of all 1s: $v \cdot v = v(1)$. By definition of the generalized limit, we have $g(v(1)) = 1$. Thus, by definition of v-scale-invariance, we should have $g(v(1)) = g(v) \cdot g(v)$. However, here, $g(v(1)) = 1$ while $g(v) \cdot g(v) = 0 \cdot 0 = 0 \ne 1$. This contradiction shows that our assumption is wrong, and it is therefore not possible to have an x-shift-invariant and v-scale-invariant generalized limit.

The proposition is proven.

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References

1. N. Chutiwongs, “The Buddha’s footprints", *Ancient Ceylon*, 1990, Vol. 10, pp. 59–116.
2. C. Cicuzza, *A Mirror Reflecting the Entire World. The Pāli Buddhapādamaṅgala or “Auspicious signs on the Buddha’s feet"*, Critical edition with English Translation, Materials for the Study of the Tripiṭaka, Vol. VI, Lumbini International Research Institute, Bangkok and Lumbini 2011.
3. M. Niwa, *Buddha’s Footprints, Pictures and Explanations: Buddhism As Seen Through the Footprints of Buddha*, Meicho Shuppan. Tokyo, 1992.
4. S. Prasopchingchana, “History and cultural heritage: past and future”, *International Journal on Humanistic Ideology*, 2013, Vol. 6, No. 1, pp. 85–103,
5. C. Stratton, *Buddhist Sculpture of Northern Thailand*, Serindia Publications, Chicago, Illinois, 2003.
6. J. S. Strong, *Relics of the Buddha*, Princeton University Press, Princeton, New Jersey, 2004.