Why Triangular Smoothing?

Shova Kuikel, Bidur Niroula, and Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso
500 W. University, El Paso, TX 79968, USA
skuikel@miners.utep.edu, bniroula@miners.utep.edu, vladik@utep.edu

Formulation of the problem. Seismic signals $x(t)$ come with a lot of high-frequency noise. To decrease the effect of this noise, geophysicists apply *smoothing*, i.e., replace $x(t)$ with $y(t) = \int x(s) \cdot a(t-s)\,ds$ for some smooth function $a(t) \ge 0$ for which $a(-t) = a(t)$ and for which, for $t > 0$, $a(t)$ decreases to 0.

Empirically, often, the most efficient smoothing comes from using a triangular function

$$a(t) = \max\left(0, 1 - \frac{|t|}{\tau}\right)$$

for some $\tau > 0$; see, e.g., [1]. In this talk, we provide a possible theoretical explanation for this efficiency.

Analysis of the problem and the resulting explanation. From the mathematical viewpoint, smooth means that the function has a derivative. However, from the practical viewpoint, if this derivative $a'(t)$ is too high, we could not call this function smooth. So, let us interpret smooth as $|a'(t)| \le M$ for some $M > 0$.

So, the problem takes the following form: on the set of all the functions $a(t)$ for which $a(t) \ge 0$ and $|a'(t)| \le M$ for all t, we need to find a function for which some quantity $E(a)$ – describing efficiency – reaches its maximum.

According to calculus, in general, the maximum of any function in a region is attained:

- either at a point where all the derivatives of this functions are 0,
- or at the border of this region.

When the region is small, it is highly improbable that the point at which all derivatives are equal to 0 lies within this region. So, with high probability, the maximum is attained at the border of the region.

When the region is determined by inequalities – as in our case – the border is when one of these inequalities becomes an equality. So, we can safely restrict ourselves to the border of the original region. To this new – smaller – region, we can apply the same argument and conclude that with high probability, the maximum is attained at the border of this new region, i.e., at a point where one more inequality becomes an equality.

We can repeat this argument again and again and conclude that with high probability, the maximum is attained when as many inequalities as possible become equalities. In our case, this means that for every t, we have either $a(t) = 0$ or $|a'(t)| = M$. For $t > 0$, the function is decreasing, so we must have $a'(t) = -M$. In this case, we have $a(t) = a(0) - M \cdot t$ – until we reach the value 0. In other words, the most effective smoothing is indeed the triangular one.

[1] S. Greer and S. Fomin, "Matching and merging high-resolution and legacy seismic images", *Geophysics*, 2018, Vol. 83, No. 2, pp. V115–V122.