Fuzzy Ideas Explain Fechner Law and Help Detect Relation Between Objects in Video

1st Olga Kosheleva
Department of Teacher Education
University of Texas at El Paso
El Paso, Texas, USA
olgak@utep.edu

2nd Vladik Kreinovich
Department of Computer Science
University of Texas at El Paso
El Paso, Texas, USA
vladik@utep.edu

3rd Ahnaf Farhan
Department of Computer Science
University of Texas at El Paso
El Paso, Texas, USA
afarhan@miners.utep.edu, farhanas@ieee.org

***Abstract*—How to find relation between objects in a video? If two objects are closely related – e.g., a computer and it mouse – then they almost always appear together, and thus, their numbers of occurrences are close. However, simply computing the differences between numbers of occurrences is not a good idea: objects with 100 and 110 occurrences are most probably related, but objects with 1 and 5 occurrences probably not, although $5-1$ is smaller than $110-100$. A natural idea is, instead, to compute the difference between re-scaled numbers of occurrences, for an appropriate nonlinear re-scaling. In this paper, we show that fuzzy ideas lead to the selection of logarithmic re-scaling, which indeed works very well in video analysis – and which also explains Fechner Law in psychology, that our perception of difference between two stimuli is determined by the difference between the logarithms of their intensities.**

***Index Terms*—fuzzy ideas, relation between objects in a video, Fechner Law**

I. Formulation of the Problem

Practical problem with which we started this research. In a video, we usually have several objects that appear from time to time. One of the things that we need to know in order to understand the video is:

- which pairs of objects are related – and if they are related, to what extent, and
- which pairs of objects are not related to all.

A seemingly natural idea. If the objects are closely related – e.g., a computer and its mouse – then in most cases, they will appear or not appear at the same time. Yes, there will be some cases when we see the computer and not the mouse – and vice versa, the mouse but not the computer – but in general, they would appear approximately the same number of times. On the other hand, if two objects are not related, then it is highly improbable that these two unrelated objects will appear the exact number of times.

It therefore seems reasonable to take the difference between the numbers of appearances of the two objects as a measure of the degree to which these objects are not related.

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395 (Center for Collective Impact in Earthquake Science C-CIES), and by the AT&T Fellowship in Information Technology.
It was also supported by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

A direct implementation of this idea and why it does not work. In a nutshell, the above idea means that the smaller the difference between the two numbers of occurrence, the more probable it is that the two objects are related. Let us explain why there is a problem with the direct implementation of the above idea. For this purpose, let us consider the following two situations.

In the first situation, we have two related objects that most frequently appear together – e.g., in 90% of the cases. In this situation, the numbers of occurrences can be, e.g., 90 and 100. The difference between these two numbers is $|100-90| = 10$, and the objects are related.

In the second situation, we have two unrelated objects that occur rarely in the video. For example, one of the two objects appeared 1 time, and another one appeared 5 times. In this case, the difference between the numbers of occurrences is $|5-1| = 4$, which is much smaller than 10 – but we can hardly make a conclusion that these two objects are related.

In this example, we have small difference for unrelated objects and a larger difference for related ones – contrary to the above idea.

Similar examples from other application areas. A similar problem appears in other application areas. For example, if we look for economically similar folks, at first glance, a seemingly reasonable idea is to use the difference between the yearly incomes as a measure of their dissimilarity. However:

- the economic difference between a poor student who gets \$20K per year and a professor who gets \$70K per year is huge, while
- the two billionaires whose annual incomes are \$2 billion and \$4 billion are economically similar.

On the other, the difference $|70-20| = 50$K in the first situation is much smaller than the 2-billion difference in the second situation.

Another example: we can easily see the difference between a dim 20W light bulb and a reasonable 40W bulb, but we will probably not see that much different between 100W and 130W bulbs, even though for the second pair, the difference is larger.

How can we modify this idea to make it work? A natural idea is to *re-scale* the numbers of occurrences before computing the difference, i.e., compute the difference:

- not between the numbers of occurrences $|a - b|$,
- but between the appropriate re-scaled numbers, i.e., the difference $|f(a)-f(b)|$ between the values $f(a)$ and $f(b)$ for an appropriate re-scaling function $f(a)$ – a function from positive numbers to positive numbers.

Which re-scaling should we choose? To utilize this idea, we need to select an appropriate function $f(a)$.

What we do in this paper. The problem of selecting a re-scaling is not mathematically well-defined, it is formulated by using imprecise terms from natural language like "appropriate". So, to solve this problem, it makes sense to use ideas from fuzzy techniques, techniques that have been designed by Lotfi Zadeh specifically for translating imprecise ("fuzzy") knowledge into precise rules; see, e.g., [1], [7], [9]–[11], [14].

In this paper, we show that these ideas indeed lead to a reasonable selection of the re-scaling function. The resulting selection indeed helps to detect relation between objects in video [2]–[5] and, as a side effect, it explains an empirical Fechner law about human perception; see, e.g., [12].

II. General Analysis of the Problem

We are interested in situations when the difference is relatively small. When the absolute value $|\Delta a|$ of the difference $\Delta a \stackrel{\text{def}}{=} b - a$ between the numbers of occurrences is huge, clearly the corresponding objects are not related.

The difficult-to-decide cases is when this difference is relatively small. So, in this paper, we will concentrate on these cases.

Let us use linearization. When the difference is small, what can we say about the difference

$$f(b) - f(a) = f(a + \Delta a) - f(a)$$

between the re-scaled values? In this case, we can use the idea actively used in physics (see, e.g., [6], [13]):

- expand this expression in Taylor series in terms of the small difference and
- only keep the first non-zero term in this expansion.

For our expression, this idea leads to

$$|f(a+\Delta a)-f(a)| \approx |(f(a)+f'(a)\cdot\Delta a)-f(a)| = F(a)\cdot|\Delta a|,$$

where we denoted $F(a) \stackrel{\text{def}}{=} f'(a)$.

What are reasonable properties of the function $F(a)$.

- The larger the value of a, the less important is the difference – e.g., for billionaires, there is practically no difference.
- On the other hand, when a is small, even a small difference is very important: for a little kid who does not have any money of his own, a dollar is a huge amount, while for a usual adult, adding \$1 would not make a big change in their happiness.

Since fuzzy technique usually deals with rules, let us formulate the above argument in terms of rules. The first bullet point can be described by the following rule:

When the value a is large, the value $F(a)$ is small.

Similarly, the second bullet point can be described by the following rule:

When the value a is small, the value $F(a)$ is large.

These two rules are actually similar, and what can we conclude based on this similarity. At first glance, the above two rules are rather different. However, a simple re-formulation makes them similar. Indeed, the intuition behind the second rule can also described in the following form:

When the value $F(a)$ is large, the value a is small.

Now, this is very similar to the first rule, the only difference is that:

- the first rule described the transformation from a to $F(a)$, while
- the reformulated second rule describes the transition from $F(a)$ to a.

We want to translate these informal rules to a precise expression. Since these two transition are described by the exact same rule, it makes sense to conclude that these two transitions are described by the same function.

In the first rule, we apply the function $F(a)$ to the value a and get $F(a)$. Similarly, in the reformulated second rule, we should apply the same function F to the value $F(a)$ and get a. In other words, we should get

$$F(F(a)) = a$$

for all a.

Terminological comment. Functions that satisfy the above property are known as *involutions*.

Remaining question. In these terms, we have the following question: *which involution should we use?*

We will show the two ways to answer this question. In the following two sections, we will use two different ideas to answer this question – and we will see that both ideas leads to the exact same conclusion.

III. First Idea

Main idea: simplicity. Our first idea is to utilize one of Zadeh's principles – the principle of simplicity: that when we have several options, we should always select the simplest one.

Comment. Simplicity is not just a philosophical idea: it can be proven that for an appropriate formalization of simplicity, this idea leads to asymptotically correct determination of the dependence from data; see, e.g., [8].

How can we define simplicity. Data processing is usually performed by computers. Intuitively:

- simple algorithms are fast to compute, while
- complex algorithms require longer computation time.

How can we gauge the computation time? In a computer, the only hardware supported operations are arithmetic operations: addition, substraction, multiplication, and division. Whatever the computer computes, it performs a sequence of arithmetic operations. For example, when we ask the computer to compute e^x, what is actually computes is the sum of the first few terms in the corresponding Taylor series:

$$e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!}.$$

It is therefore reasonable to gauge the computation time by counting the number of additions, subtractions, multiplications, and divisions in the corresponding computation – and multiplying these numbers by the time needed to perform the corresponding arithmetic operation.

We know the relation between these times:

- addition and subtraction are the fastest operations,
- multiplication is slower – since it implies several additions, and
- division is the slowest – since it involves several multiplications.

So what is the simplest involution? Can we form the function $F(a)$ by using only faster arithmetic operations, i.e., addition, subtraction, and multiplication? Not really, since by using these operations, all we get is a polynomial of a, and all polynomials tend to infinity when a increases – while we wanted a function for which for large a, the value $F(a)$ is small – in particular, for which $F(a)$ should not tend to infinity as a increases.

So, to get the desired function, we need to use at least one division. The simplest case would be if we have exactly one division and no other arithmetic operations. We start with the variable a and constants c, and we need to involve a in our computations. So, we have two choices:

$$F(a) = \frac{a}{c} \text{ and } F(a) = \frac{c}{a}.$$

- The first choice does not work, since it also leads to $F(a) \to \infty$ when a increases.
- For the second choice, the condition that $F(F(a)) = a$ means
$$\frac{c}{c/a} = a$$
and is, thus, automatically satisfied.

So, we arrive at the following conclusion.

What the first idea leads to. Our of all possible involutions $F(a)$ for which large a lead to small $F(a)$, the simplest are the functions of the type $F(a) = c/a$ for some constant c.

IV. Second Idea

Idea. Instead of looking for the simplest involution, let us look for all possible involutions that can be exactly computed, i.e., that can be computed by a finite sequence of arithmetic operations.

What does this idea mean in precise terms. One can easily check that if we start with the variable a and apply a finite number of arithmetic operations, then what we get are all *rational functions*, i.e., all ratios of two polynomials

$$R(a) = \frac{c_0 + c_1 \cdot a + \ldots + c_n \cdot a^n}{c'_0 + c'_1 \cdot a + \ldots + c'_m \cdot a^m}.$$

Indeed, each such ratio can be computed if:

- we use addition and multiplication to compute the values of the numerator and the denominator, and
- then we apply one division to compute the ratio.

Vice versa, one can easily show that:

- the variable and the constants are rational function, and
- the sum, difference, product, and ratio of two rational functions are also rational function.

Thus, by induction, any sequence of arithmetic operations results in a rational function.

So, the second idea simply means that we are looking for rational functions that are involutions. Such functions can be described by the following proposition:

Proposition. *The only rational functions $F(a)$ from positive numbers to positive numbers which are involutions and for which $F(a)$ does not tends to infinity as a increases are the function $F(a) = c/a$ for some $c > 0$.*

Comments.

- For readers' convenience, the proof of this result is placed in a special (last) section of this paper.
- So, we arrive at the same conclusion as when we used the first idea:

What the second idea leads to. The only exactly-computable (rational) convolutions $F(a)$ for which large a lead to small $F(a)$ are the functions $F(a) = c/a$ for some constant c.

V. What This Choice of the Involution Leads to

What is the resulting transformation. Once we have the fuzzy-motivated condition that the function $F(a)$ is an involution, both our ideas lead to the same result: that $F(a) = c/a$ for some constant c.

We may recall that $F(a)$ is the derivative of the desired transformation function $f(a)$: $F(a) = f'(a)$. Thus, we conclude that $f(a) = c \cdot \ln(a) + C$ for some constant C.

Of course, for computing the differences $|f(a) - f(b)|$, the constant C is irrelevant: it disappears when we compute the difference $f(a) - f(b)$. Thus, without losing generality, we can state that the resulting transformation function has the form $f(a) = c \cdot \ln(a)$ for some constant c.

This explains the Fechner Law. Fechner Law about human perception (see, e.g., [12]) states exactly this: that the perceived difference between two stimuli a and b (be they audio, visual, or others) is determined by the difference $|\ln(a) - \ln(b)|$ between the logarithms of their intensities. Thus, our fuzzy-motivated ideas explain the Fechner Law.

This helps detect relation between objects in a video. The use of logarithms also helps to solve the problem that we started with: detecting relation between objects in a video; see [2]–[5].

VI. Proof of the Proposition

Let $F(a)$ be a rational function which is an involution, i.e., for which $F(F(a)) = a$ for all positive values a, and let us assume that $F(a)$ does not tend to infinity as a increases. Let us prove that we then have $F(a) = c/a$ for some constant c.

1°. Let us first deal with the function $F(a)$ as defined for all positive real numbers.

1.1°. Let us first prove that the function $F(a)$ is one-to-one, i.e., that for different positive numbers $a \neq b$, the values $F(a)$ and $F(b)$ should be different.

Indeed, if these values were equal, i.e., if we had $F(a) = F(b)$, then we will have $F(F(a)) = F(F(b))$. However, since $F(a)$ is an involution, that would imply $a = b$. But we assumed that $a \neq b$. This contradiction shows that we cannot have $F(a) = F(b)$. Thus, $a \neq b$ indeed implies that $F(a) \neq F(b)$.

1.2°. Let us now prove that the function $F(a)$ is an onto function, i.e., that every positive number a is equal to $F(x)$ for some positive number x.

Indeed, this is true for $x = F(a)$.

1.3°. Parts 1.1 and 1.2 of this proof show that $F(a)$ is one-to-one onto function. Such functions are known as bijections.

1.4°. Every rational function is continuous everywhere it is defined, so $F(a)$ is a continuous function.

1.5°. The function $F(a)$ is a continuous bijection from the set of all positive numbers to itself. It is known that all such functions are either strictly increasing or strictly decreasing.

Let us prove this by contradiction. Since the function $F(a)$ is one-to-one, when $a < b$, we cannot have $F(a) = F(b)$, we must have either $F(a) < F(b)$ or $F(a) > F(b)$.

- Strictly increasing means that for $a < b$ we always have
$$F(a) < F(b).$$
- Strictly decreasing means that for $a < b$ we always have
$$F(a) > F(b).$$

We want to prove that if we have $F(a) > F(b)$ for some $a < b$, then we cannot have $F(a') > F(b')$ for some $a' < b'$. Indeed, for every $\alpha \in (0, 1)$, we can conclude that $\alpha \cdot a < \alpha \cdot a'$ and $(1 - a) \cdot b < (1 - \alpha) \cdot b'$ and thus, $\alpha \cdot a + (1 - \alpha) \cdot b < \alpha \cdot b + (1 - \alpha) \cdot b'$. For $\alpha = 0$ and $\alpha = 1$, this inequality is also true – it corresponds to $a < b$ and $a' < b'$.

Let us now consider the function that maps $\alpha \in [0, 1]$ into the difference
$$F(\alpha \cdot a + (1 - \alpha) \cdot a') - F(\alpha \cdot b + (1 - \alpha) \cdot b').$$

This function is a composition of continuous functions and is, therefore, continuous itself.

- When $\alpha = 0$, this difference is equal to $F(a') - F(b')$ and is, therefore, positive.
- When $\alpha = 1$, this difference is equal to $F(a) - F(b)$ and is, therefore, negative.

Thus, by the Intermediate Value Theorem for continuous functions, there exists an $\alpha \in (0, 1)$ for which this difference is equal to 0, i.e., for which $\alpha \cdot a + (1 - \alpha) \cdot b < \alpha \cdot b + (1 - \alpha) \cdot b'$ but
$$F(\alpha \cdot a + (1 - \alpha) \cdot a') = F(\alpha \cdot b + (1 - \alpha) \cdot b').$$

This contradicts to the fact that the function $F(a)$ is one-to-one.

Thus, the function $F(a)$ is indeed either strictly increasing or strictly decreasing.

1.6°. We assumed that the function $F(a)$ is not strictly increasing, thus it must be strictly decreasing. Since this function is onto, values close to 0 must map to values close to infinity and vice versa, i.e., we must have $F(a) \to 0$ as $a \to \infty$.

2°. Let us now deal with the natural extension of a function $F(a)$ to all complex numbers.

2.1°. Every rational function is analytical, so our involution is analytical too. In particular, we can naturally extend it to all complex values z. Of course, for some z, the denominator can be 0, so for these z, we will have $F(z) = \infty$. The composition of rational functions is also rationl, so the function $F(F(z))$ is also analytical.

2.2°. It is known that if two analytical functions are equal on an infinite set that contains a limit point, then these two functions are equal for all z.

In our case, the functions $F(F(z))$ and z are equal for all positive real numbers z. The set of all positive numbers clearly contains many limit points, so we conclude that $F(F(z)) = z$ for all complex values z.

2.3°. In complex domain, each polynomial of each degree n – including the numerator and the denomination of the expression $F(z)$ – have exactly n roots $z_1, \ldots, z_n$ – if we count roots with multiplicity the corresponding number of times. Thus, each polynomial can be represented as a product
$$c \cdot (z - z_1) \cdot \ldots \cdot (z - z_n).$$

So, the rational function $F(z)$ can be represented as the ratio of two such products:
$$F(z) = \frac{c \cdot (z - z_1) \cdot \ldots \cdot (z - z_n)}{c' \cdot (z - z'_1) \cdot \ldots \cdot (z - z'_m)}.$$

We can simplify this expression if we divide both numerator and denominator by c'. Then the constant c' in the denominator disappears.

Also, if both the numerator and the denominator contain the same difference $z - z_i$, then we can divide both numerator and denominator by this difference. Thus, we can conclude that the function $F(z)$ has the following form:

$$F(z) = \frac{c \cdot (z - z_1) \cdot \ldots \cdot (z - z_n)}{(z - z'_1) \cdot \ldots \cdot (z - z'_m)},$$

where all the values z_i are different from all the values z'_j.

2.4°. Let us show that all the roots in the numerator are equal to each other.

We will prove this by contradiction. Suppose that we have $z_i \neq z_j$ for some $i \neq j$. Then we would have $F(z_i) = F(z_j) = 0$. Thus, $F(F(z_i)) = F(F(z_j))$.

On the other hand, since $F(z)$ is an involution, we should have $F(F(z_i)) = z_i \neq F(F(z_j)) = z_j$ – which contradicts to our conclusion that $F(F(z_i)) = F(F(z_j))$. This contradiction shows that the roots in the numerator cannot be different.

2.5°. Similarly, we can prove that all the roots in the denominator are equal to each other.

We can also prove it by contradiction. Indeed, suppose that we have $z'_i \neq z'_j$ for some $i \neq j$. Then we would have $F(z'_i) = F(z'_j) = \infty$. Thus, $F(F(z'_i)) = F(F(z'_j))$.

On the other hand, since $F(z)$ is an involution, we should have $F(F(z'_i)) = z'_i \neq F(F(z_j)) = z'_j$. The contradiction shows that these roots cannot be different.

2.6°. Due to Parts 2.4 and 2.5 of this proof, the function $F(z)$ has the form

$$F(z) = \frac{c \cdot (z - z_1)^n}{(z - z'_1)^m},$$

for some $n \geq 0$ and $m \geq 0$.

2.7°. Since we want $F(a) \to 0$ as $a \to \infty$, we must have $m > n$. Since $n \geq 0$, this means that $m \geq 1$, so there is some value z'_1.

For $z = z'_1$, we have $F(z'_1) = \infty$. We know that $F(\infty) = 0$, thus $F(F(z'_1)) = 0$. Since the function $F(z)$ is an involution, we have $F(F(z'_1)) = z'_1$, thus $z'_1 = 0$. So, the function $F(z)$ has the form

$$F(z) = \frac{c \cdot (z - z_1)^n}{z^m},$$

for some $m > 0$. In this case, $F(0) = \infty$.

2.8°. What happens if $n > 0$? In this case, we will have

$$F(z_1) = 0.$$

We know, however, that $F(0) = \infty$, so

$$F(F(z_1)) = F(0) = \infty.$$

This contradicts to the involution equality $F(F(z_1)) = z_1$. This contradiction shows that we cannot have $n > 0$, so $n = 0$. Thus, the function $F(z)$ has the form

$$F(z) = \frac{c}{z^m}.$$

2.9°. For the above expression, the involution requirement takes the form

$$F(F(z)) = \frac{c}{(c/z^m)^m} = \frac{c}{c^m/z^{m^2}} = c^{1-m} \cdot z^{m^2} = z.$$

This equality must be satisfied for all z, so we must have $m^2 = 1$. Thus – since m is a positive number – we must have $m = 1$. In this case, the desired involution equality is always satisfied,

So, we indeed conclude that $F(z) = c/z$ for some $c > 0$. The proposition is proven.

References

[1] R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

[2] A. Farhan, *Context-Aware Temporal Embeddings for Text and Video Data*, Ph.D. Dissertation, University of Texas at El Paso, December 2023.

[3] A. Farhan, R. Camacho Barranco, M. Akbar, and M. S. Hossain, "Temporal word embedding with predictive capability", *Knowledge and Information Systems*, 2023, Vol. 65, pp. 5159–5194.

[4] A. Farhan, R. Camacho Barranco, M. S. Hossain, and M. Akbar, "Diffusion-based temporal word embeddings", *Proceedings of the Workshop on Scientific Document Understanding co-located with 35th AAAI Conference on Artificial Inteligence AAAI 2021*, February 8–9, 2021.

[5] A. Farhan and M. S. Hossain, "Vizobj2vec: Contextual representation learning for visual objects in video-frames", *Proceedings of the 2020 IEEE International Conference on Big Data*, Atlanta, Georgia, USA, December 10–13, 2020, pp. 689–698.

[6] R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.

[7] G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.

[8] M. Li and P. Vitanyi, *An Introduction to Kolmogorov Complexity and Its Applications*, Springer, Cham, Switzerland, 2019.

[9] J. M. Mendel, *Uncertain Rule-Based Fuzzy Systems: Introduction and New Directions*, Springer, Cham, Switzerland, 2017.

[10] H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.

[11] V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.

[12] S. K. Reed, *Cognition: Theories and application*, SAGE Publications, Thousand Oaks, California, 2022.

[13] K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2021.

[14] L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.