Somewhat Surprisingly, (Subjective) Fuzzy Technique Can Help to Better Combine Measurement Results and Expert Estimates into a Model with Guaranteed Accuracy: Digital Twins and Beyond

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Abstract To understand how different factors and different control strategies will affect a system – be it a plant, an airplane, etc. – it is desirable to form an accurate digital model of this system. Such models are known as digital twins. To make a digital twin as accurate as possible, it is desirable to incorporate all available knowledge of the system into this model. In many cases, a significant part of this knowledge comes in terms of expert statements, statements that are often formulated by using imprecise ("fuzzy") words from natural language such as "small", "very possible", etc. To translate such knowledge into precise terms, Zadeh pioneered a technique that he called fuzzy. Fuzzy techniques have many successful applications; however, expert statements are subjective; in contrast to measurement results, they do not come with guaranteed accuracy. In this paper, we show that by using fuzzy techniques, we can translate imprecise expert knowledge into precise probabilistic terms – thus allowing to combine this knowledge with measurement results into a model with guaranteed accuracy.

1 Formulation of the problem

Need for digital twins. It is desirable to predict how a building, an airplane, a ship, or any other structure will behave under different circumstances – and thus, e.g., to

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make sure that they will remain functional under expected extreme conditions. It is desirable to predict what will happen if we close some roads for repairs, etc. – and thus, to select the schedule of needed road maintenance that minimally disturbs the traffic.

In all these cases, we need to have a digital model of the corresponding system, a model which is as accurate as possible, so that it ideally, the model and the system should be as close to each other as twins. Such models are therefore called *digital twins*.

Need to take into account expert knowledge. To make the model of a system as accurate as possible, we need to incorporate all available information about this system into this model. Usually, a significant part of this information comes in precise terms:

- in terms of the values of some quantities characterising the system, values that can be obtained by measuring these quantities,
- equation relating these quantities and/or describing how the values of these quantities charge with time, etc.

In some cases, this precise knowledge is sufficient to build an accurate model of a system: for example, Newton's equations accurately predict the positions of all celestial bodies hundreds of years ahead, so that, e.g., we know on what days there will be a solstice in the 30th century.

However, in many other cases, the precise knowledge is not sufficient. In such cases, we do not have a fully automated controllers for these systems: we still need expert controllers. The experience of these controllers and other experts provide additional knowledge. The most extreme case is probably medicine, where human doctors are still irreplaceable.

It is therefore necessary to incorporate the expert knowledge into our models.

How can we avoid subjectivity when we incorporate expert knowledge into a model. Experts have biases, expert estimates come with uncertainty that varies from one expert to another. In other words, values produced by experts are somewhat subjective.

At first glance, it may seem that if we add expert knowledge into a model, the model becomes somewhat subjective – and thus, less reliable. However, the situation is not so bad if we take into account that the ultimate sources of precise information – measuring instruments – also have individual biases and varying degrees of uncertainty.

To take that into account, we *calibrate* each measuring instrument – by comparing the values measured by this instrument and the values measured in the same situation by a much more accurate instrument. In measurement theory, such a much more accurate instrument is called *standard*; see, e.g., [6].

In each such measurement, the measured quantity has some actual value x (that we do not know). By applying our measuring instrument to this quantity, we get a value $\widetilde{x}$ which is, in general, different from x: there is a *measurement error* $\Delta x \stackrel{\text{def}}{=} \widetilde{x} - x$. By applying the standard measuring instrument to this same physical quantity, we get a different measurement result $\widetilde{X}$. The fact that the standard measuring instrument is much more accurate means that its measurement error is much smaller: $\Delta X \stackrel{\text{def}}{=} \widetilde{X} - x \ll \Delta x$. The difference $\widetilde{x} - \widetilde{X}$ between the two measurement results can be represented in the following equivalent way is we add and subtract x and group the terms:

$$\widetilde{x} - \widetilde{X} = \widetilde{x} - \widetilde{X} + x - x = (\widetilde{x} - x) - (\widetilde{X} - x) = \Delta x - \Delta X. \tag{1}$$

Since $\Delta X \ll \Delta x$, we can safely ignore the term ΔX in the formula (1) and conclude that, with good accuracy, $\Delta x \approx \widetilde{x} - \widetilde{X}$.

In this approximation, by comparing the measurement results of two instruments, we get one of the possible values of the measurement error Δx of the original measuring instrument. If we repeat this procedure many times, with different objects, we get a sample of different values Δx – and based on this sample, we can find the probability distribution of the measurement errors.

A natural idea is to apply the same approach to a human expert:

- we ask the human expert to provide an estimate $\widetilde{x}$ for the value of some quantity,
- we compare this estimate withe the result $\widetilde{X}$ of accurately measuring this quantity, and
- we use the sample formed by the differences $\widetilde{x} - \widetilde{X} \approx \Delta x$ to determine the probability distribution of the expert's estimation errors $\Delta x = \widetilde{x} - x$.

Once we know this distribution, we can use each expert as an additional measuring instrument – with not more subjectivity than when we use sensors.

Can we do better? The main idea of dealing with expert knowledge is to treat an expert as a kind of a measuring instrument. However, from the viewpoint of information, there is an important different between experts and measuring instruments.

- A measuring instrument only produces a single number $\widetilde{x}$.
- In contrast, in many cases, an expert not only produces an estimate $\widetilde{x}$, he/she can also describe how possible are different deviation of this estimate from the actual value of the estimated quantity. For example, if the estimate is 1.0, the expert may add that a value 0.9 is very possible, the value 0.8 is somewhat possible, and the value 0.7 is hardly possible. The expert may say that the difference between his/her estimate and the actual value is small, etc.

How can we incorporate this subjective additional information into a model with guaranteed accuracy? This is what we discuss in this paper.

2 Analysis of the problem

How is imprecise expert information described now: a brief reminder about fuzzy techniques. The problem with the expert information about uncertainty – of the type described above – is that it is described not in precise terms, but by using imprecise ("fuzzy") words from natural language, like "very possible", "somewhat possible", "small", etc. The need to take this knowledge into account when designing computer-based algorithms – e.g., control strategies – was emphasized in the 1960s by Lotfi Zadeh who came up with a special technique for translating this knowledge into precise terms, a technique that he called *fuzzy*; see, e.g., [1, 2, 3, 4, 5, 7].

Zadeh's main idea was to as the expert to assign, to each possible number, a degree – from the interval $[0,1]$ – to which the expert believes it be possible that this number is the actual value of the estimated quantity. Here:

- degree 1 means that the expert absolutely believes that this number is possible,
- degree 0 means that the expert absolutely believes that this number is *not* possible,
- and degrees between 0 and 1 means that the expert is not 100% sure.

Based on the expert's answers to these questions, we get a function $\mu(x)$ that assigns, to eah value x, the corresponding degree. This function is known as a *membership function*, or, alternatively, as a *fuzzy set*.

In applications, we not only deal with basic statements $A, B, \ldots$ Expert knowledge is often formulated in terms of logical combinations of these statements such as $A\,\&\,B$, $A \vee B$, $\neg A$, etc. In many practical situations:

- all we know about these statements is their degree of confidence a, b, $\ldots$, and
- we need to estimate the expert's degree of confidence in the corresponding logical combinations based on these known degrees.

The corresponding algorithms $f_\&(a,b)$, $f_\vee(a,b)$, $f_\neg(a)$ – known as *logical operations* – form what is known as *fuzzy logic*.

There are many such operations. For example, the most widely used "or"-operation is $f_\vee(a,b) = \max(a,b)$. This the operation that we will use in this paper – but there are others, e.g., $f_\vee(a,b) = a + b - a \cdot b$. In the following text, we will explain why we use this specific operation.

But fuzzy sets are subjective, so what can we do? Fuzzy sets are very useful when we need, e.g., to transform natural-language expert rules into a precise control strategy. However, these estimates are subjective – so we cannot directly incorporate them into a model with guaranteed accuracy. So what can we do?

A natural idea: using calibration. We have already encountered the situation when we need to incorporate subjective information into a model with guaranteed accuracy – this is what we described in the previous section. There, the solution was to calibrate the measuring instrument – and, similarly, to calibrate the expert.

It is therefore natural to try a similar idea her. Let us describe how this can be done. For this purpose, let us recall an alternative way of representing fuzzy sets – a way that is used for computations related to fuzzy sets.

α-cuts: a computational-friendly representation of a fuzzy set. For computations with fuzzy sets, it is convenient to generate *α-cuts* corresponding to different values $\alpha \in [0,1]$:

- For $\alpha > 0$, the α-cut is defined as $\mathbf{x}(\alpha) \stackrel{\text{def}}{=} \{x : \mu(x) \geq \alpha\}$.
- For $\alpha = 0$, the corresponding α-cut is defined as $\mathbf{x}(0) \stackrel{\text{def}}{=} \overline{\{x : \mu(x) > 0\}}$, where $\overline{S}$ means the closure of the set S (i.e., the result of adding, to the set S, all its limit points).

Once we know all the α-cuts, we can reconstruct the original membership function as $\mu(x) = \sup\{\alpha : x \in \mathbf{x}(\alpha)\}$.

For most types of fuzzy information, the corresponding degree $\mu(x)$ first increases, then decreases. In this case, each α-cut is an interval $\mathbf{x}(\alpha) = [\underline{x}(\alpha), \overline{x}(\alpha)]$.

What is the meaning of an α-cut: let us brainstorm. By definition of the α-cut, for each α, and for each number x outside the α-cut interval $\mathbf{x}(\alpha)$, the expert's degree of confidence $\mu(x)$ that this number is a possible value of the estimated quantity is smaller than α. What is the expert's degree of confidence that "*some* number x outside this interval is a possible value of the estimated quantity"?

The statement S whose degree we want to estimate means that:

- either one of these numbers x_1 is a possible value of the estimated quantity
- *or* another of these numbers x_2 is a possible value of the estimated quantity, etc.

In other words, the statement S is, in effect, an "or"-combination of statements corresponding to individual values x. There are infinitely many numbers outside the α-cut interval, so we have an "or"-combination of infinitely many such statements. If we use the max-operation $f_\vee(a,b)$, then we simply take the maximum (or, to be more precise, supremum) of all the numbers which are smaller than α – which is, of course, exactly α.

This, by the way, explains why we use maximum and not any other "or"-operation like $f_\vee(a,b) = a + b - a \cdot b$: one can easily check that if we apply this operation to many similar numbers, the results start tending to 1, and in the limit when we take all infinite numbers outside the α-cut interval into account, we get a meaningless degree 1. Same thing happens with many other known "or'-operations.

So, the expert's degree of confidence that one of the numbers x outside the α-cut interval is a possible value of the estimated quantity is α. Thus, the expert's degree of confidence in the opposite statement – that none of the numbers outside the α-cut interval is a possible value of the estimated quantity, i.e., that the actual value is inside the α-cut interval – is equal to $f_\neg(\alpha)$. So, we arrive at the following conclusion.

What is the meaning of an α-cut: conclusion. For each α, we can conclude that the expert's degree of confidence that the actual value x is in the α-cut interval $[\underline{x}(\alpha), \overline{x}(\alpha)]$ is equal to $f_\neg(\alpha)$.

What we need to know to make the expert statement more objective. Fuzzy techniques use "degree of confidence", which is a very subjective value. A natural objective counter-part of this subjective notion is probability. So, what we need is to come up with *probabilities* that x is in each α-cut.

How can we transform subjective degrees into objective probabilities: a natural idea. As we have mentioned, a natural idea of transforming subjective values into objective probabilities is by using calibration. Specifically, we ask the expert to provide the membership function for many different cases. Based on these membership functions, we come up with the corresponding α-cut intervals. For all these cases, we also use a standard measuring instrument to measure the actual value of the corresponding quantity.

Then, for each α, we can count in what proportion of these cases the actual value x was in the corresponding α-cut interval. This proportion $p(\alpha)$ is, thus, an estimate for the (objective) probability that the actual value x is in this expert's α-cut interval.

We are almost done. After the above calibration, we get the function $p(\alpha)$ corresponding to this particular expert.

Then, for each new expert's estimate, we can conclude that, based on this expert's opinion, the probability that the actual value likes in the corresponding α-cut interval is equal to $p(\alpha)$.

This *almost* determines the probability distribution, but not yet fully. For example, for two close α-cut intervals $[\underline{x}(\alpha),\overline{x}(\alpha)] \supseteq [\underline{x}(\alpha+\Delta\alpha),\overline{x}(\alpha+\Delta\alpha)]$, we know the probability $p(\alpha)$ of the larger interval and the probability $p(\alpha+\Delta\alpha)$ of the smaller interval. Thus, we know that the probability that the actual value x is in the difference between these two intervals – which consists of two small intervals $[\underline{x}(\alpha),\underline{x}(\alpha+\Delta\alpha)]$ and $[\overline{x}(\alpha+\Delta\alpha),\overline{x}(\alpha)]$ – is equal to the difference $p(\alpha)-p(\alpha+\Delta\alpha)$ between these two probabilities.

However, this information does not tell us what is the probability of each of the two small intervals – it only provides the sum of these two small-interval probabilities. To have this information, let us use another natural idea.

Final idea. To explain this idea, let us recall that the most frequently experiences probability distribution is normal (Gaussian). A specific feature of this distribution is that its probability density $f(x)$ is always positive. In principle, all numbers are possible – but, of course, very large numbers are highly improbable. In practice, we ignore such practically improbable events and only consider values x that are reasonably probable.

The degree to which each value x is probable is described by the value $f(x)$ of the corresponding probability density function. From this viewpoint, if $f(x) \le f(y)$ and we consider x to be reasonably probable, then y is even more probable – and thus, also has to be considered reasonably probable. So, when we want to select a set to which x belongs to some degree of certainty, a natural idea is to select some threshold f_0 and consider all the numbers x fior which $f(x) \ge f_0$.

It is reasonable to assume that the expert-based intervals $[\underline{x}(\alpha),\overline{x}(\alpha)]$ are obtained in exactly this way, i.e., that the probability density is exactly the same on both ends of this interval. Thus, the probability $p(\alpha)-p(\alpha+\Delta\alpha)$ is distributed between the two small intervals – $[\underline{x}(\alpha),\underline{x}(\alpha+\Delta\alpha)]$ and $[\overline{x}(\alpha+\Delta\alpha),\overline{x}(\alpha)]$ – proportionally to their lengths.

Now we are ready. Now, we are finally ready to formulate the desired calibration-based transformation of the experts fuzzy opinion into an objective probability distribution – that we can incorporate into a model with guaranteed accuracy.

3 How to transform subjective expert estimates into an objective probability distribution

First stage: testing an expert – ideal case. First, for several (N) cases, we ask the expert not only provide a single estimate $\widetilde{x}$ for the given quantity, but to also describe, for every real number x, the degree $\mu(x)$ to which this number x can be the actual value of the estimated quantity.

Based on the expert-produced degrees, for each value $\alpha \in [0,1]$, we form the α-cut $\mathbf{x}(\alpha) = [\underline{x}(\alpha), \overline{x}(\alpha)]$:

- $\mathbf{x}(\alpha) = \{x : \mu(x) \geq \alpha\}$ when $\alpha > 0$ and
- when $\alpha = 0$, we compute $\mathbf{x}(0) = \overline{\{x : \mu(x) > 0\}}$.

Testing an expert – practical solution. Of course, there are infinitely many real numbers, we can only ask the expert about finitely many of them $x_1 < x_2 < \ldots < x_n$. To estimate the values $\mu(x)$ corresponding to intermediate values $x \in (x_i, x_{i+1})$, we can, for example, use linear interpolation

$$\mu(x) = \mu(x_i) + \frac{x - x_i}{x_{i+1} - x_i} \cdot (\mu(x_{i+1}) - \mu(x_i)).$$

Similarly, there are infinitely many real numbers α on the interval $[0,1]$. We cannot perform computations infinitely many times. So, in practice, we can:

- select several possible values $0 = \alpha_1 < \alpha_2 < \ldots < \alpha_n = 1$ – for example, values $0 < 0.1 < 0.2 < 0.3 < \ldots < 0.9 < 1$, and
- only compute the α-cuts for these selected value α, i.e., α-cuts

$$[\underline{x}(\alpha_1), \overline{x}(\alpha_1)] \supseteq [\underline{x}(\alpha_2), \overline{x}(\alpha_2)] \supseteq \ldots \supseteq [\underline{x}(\alpha_n), \overline{x}(\alpha_n)].$$

For the endpoints of these α-cuts, we have a natural order:

$$\underline{x}(\alpha_1) \leq \underline{x}(\alpha_2) \leq \ldots \leq \underline{x}(\alpha_n) \leq \overline{x}(\alpha_n) \leq \overline{x}(\alpha_{n-1}) \leq \ldots \leq \overline{x}(\alpha_2) \leq \overline{x}(\alpha_1).$$

Second stage: comparing expert's estimates with actual values. For all these cases, we compare the expert estimates with the actual value – or, to be more precise, with the value x provided by the standard measuring instrument.

For each of the selected values α_i, we count the number N_i of cases in which the actual value x was inside the corresponding α-cut interval $[\underline{x}(\alpha_i), \overline{x}(\alpha_i)]$. Based on these counts, we compute the ratios $p(\alpha_i) \stackrel{\text{def}}{=} n_i/N$.

Final stage: generating the resulting probability distribution. As a result, we get the following probability density function $f(x)$.

- For values $x \in [\underline{x}(\alpha_i), \underline{x}(\alpha_{i+1})]$, we have

$$f(x) = \frac{p(\alpha_i) - p(\alpha_{i+1})}{(\underline{x}(\alpha_{i+1}) - \underline{x}(\alpha_i)) + (\overline{x}(\alpha_i) - \overline{x}(\alpha_{i+1}))}.$$

- For values $x \in [\underline{x}(\alpha_n), \overline{x}(\alpha_n)]$, we take

$$f(x) = \frac{p(\alpha_n)}{\overline{x}(\alpha_n) - \underline{x}(\alpha_n)}.$$

- For values $x \in [\overline{x}(\alpha_{i+1}), \overline{x}(\alpha_i)]$, we have

$$f(x) = \frac{p(\alpha_i) - p(\alpha_{i+1})}{(\underline{x}(\alpha_{i+1}) - \underline{x}(\alpha_i)) + (\overline{x}(\alpha_i) - \overline{x}(\alpha_{i+1}))}.$$

Mathematical comment. In the limit, when the differences $\alpha_{i+1} - \alpha_i$ tend to 0, the above formulas take the form

$$f(x) = \frac{\dfrac{dp}{d\alpha}}{\dfrac{d\underline{x}}{d\alpha} - \dfrac{d\overline{x}}{d\alpha}},$$

where α is the value for which either $\underline{x}(\alpha) = x$ or $\overline{x}(\alpha) = x$ – i.e., the value $\alpha = \mu(x)$.

A brief conclusion. We have shown that, somewhat surprisingly, fuzzy techniques – which usually reflect a somewhat subjective expert opinion – can help to transform these estimates into an objective probabilistic form and thus, help to better combine measurement results and expert estimates into a model with guaranteed accuracy.

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