Why Two Fish Follow Each Other but Three Fish Form a School: A Symmetry-Based Explanation

Shahnaz Shahbazova, Olga Kosheleva, and Vladik Kreinovich

Abstract
Recent experiments with fish has shown an unexpected strange behavior: when two fish of the same species are placed in an aquarium, they start following each other, while when three fish are placed there, they form (approximately) an equilateral triangle, and move in the direction (approximately) orthogonal to this triangle. In this paper, we use natural symmetries – such as rotations, shifts, and permutation of fish – to show that this observed behavior is actually optimal. This behavior is not just optimal with respect to one specific optimality criterion, it is optimal with respect to any optimality criterion – as long as the corresponding comparison between two behaviors does not change under rotations, shifts, and permutations.

1 Introduction

Formulation of the problem. A recent research [1, 11] showed that, when we place several fish of the same species in an aquarium, then:
• if there are only two fish, they follow each other, but
• if there are three fish, they form a school: they place themselves in positions
  forming (approximately) an equilateral triangle, and move in the direction (ap-
  proximately) orthogonal to this triangle.

How can we explain this phenomenon?

What we do in this paper. In this paper, we provide a natural symmetry-based
explanation for this phenomenon. Namely, we show that the observed behavior is
optimal with respect to all optimality criteria that are invariant with respect to natural
symmetries: spatial rotations, shifts, and permutations of the fish.

Comment. In this paper, we will not select any specific optimality criterion, we will
not specify any numerical model of the fish behavior. Instead, we will start with a
kind-of qualitative natural-language description of the situation – namely, the one
described in the previous paragraph. Then, we will show how this natural-language
description can be translated into a precise result. In this sense, what we are doing is
very similar to what Lotfi Zadeh did when he invented fuzzy techniques [2, 4, 5, 7, 8,
10]. While our techniques are different from what Zadeh used, they still fall under
the general rubric of translating natural-language descriptions into precise terms,
rubric that can be described as – in very general sense – fuzzy.

The structure of this paper. In Section 2, we describe a symmetry-related mathe-
matical result that we will use in our explanation. In Section 3, we will apply this
result to explain the motion of a pair of fish. In Section 4, we apply this result to
explain the location and motion of three fish.

2 Symmetry-related mathematical result

What do we mean by an optimality criterion. In general, we have the set \( A \) of
alternatives, and we want to select the optimal (best) one, i.e., the one which is
better than – or of the same quality as – every other alternative. Let us denote the
relation “better than or of the same quality as” by \( \geq \), so that \( a \geq b \) would mean that
\( a \) is better than \( b \) or of the same quality as \( b \). In these terms, the optimal alternative
\( a_{\text{opt}} \) is the one for which \( a_{\text{opt}} \geq a \) for all \( a \).

Clearly, each alternative is of the same quality as itself, so we have \( a \geq a \). Rela-
tions satisfying this property are known as reflexive. Also, if \( a \) is better or of the same
quality as \( b \), and \( b \) is better or of the same quality as \( c \), then \( a \) is clearly either better
or of the same quality as \( c \). Relations satisfying this property are called transitive.
Thus, by an optimality criterion, we will mean a reflexive and transitive relation.
Such relations are known as pre-orders. So, we arrive at the following definition.

Definition 1. Let \( A \) be a set. Elements of this set will be called alternatives. By an
optimality criterion on the set \( A \), we mean a binary relation \( \geq \) that satisfies the
following two properties:

- for all \( a \), we have \( a \geq a \), and

• for all $a, b$, and $c$, if $a \geq b$ and $b \geq c$, then $a \geq c$.

We say that the alternative $a_{\text{opt}}$ is optimal if $a_{\text{opt}} \geq a$ for all $a \in A$.

**We will only consider final optimality criteria.** Our ultimate goal is to select a single alternative. So, if a current optimality criterion has several equally good optimal alternatives, this means that this criterion is not final: we can use this non-uniqueness to optimize something else.

For example, if we select the fastest algorithm for solving some problem, and several different algorithms have the same average computation time, we can select, among them, the one that has the smallest worst-case computation time. If we still have several equally good algorithms, we can use the remaining non-uniqueness to optimize something else.

At the end, we should end up with a final optimality criterion for which there is exactly one optimal alternative.

**Definition 2.** We say that an optimality criterion is final if there is exactly one alternative that is optimal with respect to this criterion.

**Natural symmetries.** From the physical viewpoint, there are often some transformations that do not change the relative quality of alternatives. For example, if one dish tastes better than another, this relation does not change if we turn and/or shift the table with the taster. In general, let $G$ denotes the set of all such transformations.

If a transformation does not change the relation, the inverse transformation should not change it either: if we move a person 100 meters North, then moving the same person 100 meters back South should not affect his/her taste. Similarly, if each of the two transformations does not change the relation, then their composition – when we first apply the first transformation and then the second transformation – also should not change the relation. Sets of transformations that contain inverse and composition are known as transformation groups. So, we arrive at the following definition.

**Definition 3.** Let $A$ be a set. By a transformation group, we mean a set $G$ of functions $g : A \to A$ for which:

• if the function $g$ is in the set $G$, then the inverse function $g^{-1}$ exists and is also in the set $G$;
• if the functions $f$ and $g$ are in the set $G$, their composition $g(f(a))$ is also in the set $G$.

We say that the optimality criterion $\geq$ is $G$-invariant if for all $g \in G$ and for all $a, b \in G$, we have $a \geq b$ if and only if $g(a) \geq g(b)$.

**Terminological comment.** In physics – where invariance is one of the main tools – transformations that keep some things invariant are called symmetries; see, e.g., [3, 9]. In line with this, we will also call such transformations symmetries.

**The result that we will use.** Now, we are ready to formulate the mathematical result that we will use to explain the behavior of the fish. To formulate this result, we need to introduce one more definition.
Definition 4. Let $A$ be a set, and let $G$ be a transformation group on $A$. We say that an element $a \in A$ is $G$-invariant if for all $g \in G$, we have $g(a) = a$.

Proposition. (see, e.g., [6]) Let $\geq$ be a final $G$-invariant optimality criterion $\geq$, then its optimal alternative $a_{\text{opt}}$ is $G$-invariant.

Proof. To prove this result, we need to prove that, for each $g \in G$, we have $g(a_{\text{opt}}) = a_{\text{opt}}$. Indeed, by definition of an optimal alternative, $a_{\text{opt}}$ is better than or of the same quality as any other alternative. In particular, for each $a$, we have $a_{\text{opt}} \geq g^{-1}(a)$. Since the optimality criterion $\geq$ is $G$-invariant, we can conclude that $g(a_{\text{opt}}) \geq g(g^{-1}(a))$. By the definition of an inverse function, we always have $g(g^{-1}(a)) = a$, so we conclude that $g(a_{\text{opt}}) \geq a$ for all $a \in A$.

By the definition of an optimal alternative, this means that the alternative $g(a_{\text{opt}})$ is optimal. But our optimality criterion is final, which means that there is only one optimal alternative. Thus, the two optimal alternatives $a_{\text{opt}}$ and $g(a_{\text{opt}})$ cannot be different, they must be equal: $g(a_{\text{opt}}) = a_{\text{opt}}$.

The statement is proven.

3 Case of two fish

Location and its symmetries. Whatever two locations the two fish select to place themselves in, these two points form a line. One can see that the this 2-point spatial configuration is not invariant with respect to any shifts, but it is invariant with respect to several rotations. We can list all the rotations that keep this spatial configuration invariant:

- all rotations around the fish-connecting line; these rotations keep both locations intact, and
- all 180 degree rotations around a different line – a line which passes through the midpoint between the locations and with is orthogonal to the fish-connecting line; these rotations swap the two locations.

How to describe possible motions. At first glance, it seems that to describe the direction of motion of this spatial configuration, we need to describe the unit vector $e$ in the direction of this motion. This would have been true if we considered a motion with a target destination – e.g., when the fish are pursued by a predator and try to reach a safe zone, which the predator cannot penetrate.

However, in the experiments that we are trying to explain, we are not talking about a clearly time-directed motion, we are talking about moving in circles. In this case, it should not matter whether we consider motions forward in time or the same motions viewed backward in time. Backward in time simple means that we reverse the direction of all velocities, i.e., consider the vector $-e$ instead of the vector $e$. From this viewpoint, what we want to describe is not so much a unit vector, but rather the direction, the pair $(e, -e)$ consisting of two opposite unit vectors.
Why Two Fish Follow Each Other but Three Fish Form a School

Which motion is optimal. It is reasonable to assume that the relative quality of different motions should not change under possible rotations. Thus, by the main result from Section 2, the optimal motion – i.e., to be more precise, the optimal dynamic configuration consisting of the fish locations and of the motion-related pair \((e, -e)\) – should be invariant with respect to all the rotations with respect to which the initial spatial configuration is invariant.

One can easily see that if the vector \(e\) is not parallel to the fish-connecting line, then the corresponding dynamic configuration is no longer invariant with respect to all the rotations around this line – since each such rotation will change the direction of the vector \(e\). Thus, the only invariant dynamic configuration is the one in which the vector \(e\) is parallel to the fish-connecting line, i.e., when fish follow each other.

Since, as we have proved, the optimal motion should lead to an invariant dynamic configuration, this means that the optimal motion is exactly the motion in which the fish follow each other – which is exactly what was observed.

4 Case of three fish

How to describe possible locations. To describe the locations of three fish, we need to select three spatial points \((x_1, x_2, x_3)\). As we have mentioned, rotations and shifts should not change the relative quality of different spatial locations. So, instead of a single triple of points, it make sense to consider, as alternatives, the sets \(s\) of all possible locations that are obtained from a triple \((x_1, x_2, x_3)\) by all possible shifts and rotations.

Which locations are optimal? Which of these sets \(s\) is optimal? Since we are talking about similar fish, it should not matter which of them we consider fish number 1 and which fish number 2 – the relative quality of different spatial configurations should not change. In other words, the optimality criterion should be invariant with respect to all possible permutations of fish.

According to our main result, this means that the optimal location-describing alternative \(s\) should also be invariant under all permutations. This means, for example, that if we rename fish 1 and 3, then the resulting triple \((x_3, x_2, x_1)\) should belong to the same optimal set \(s\), i.e., it can be obtained from the original triple \((x_1, x_2, x_3)\) by shifts and rotations. Shifts and rotations do not change distance between points, so we conclude that the distance \(d(x_3, x_2)\) between the points \(x_3\) and \(x_2\) should be equal to the distance \(d(x_1, x_2)\) between the locations of similar fish in the original triple. In other words, two sides of the triangle formed by the three fish should be equal.

By considering a different permutation, we can conclude that the third side should also be equal to the other two sides – so the optimal spatial configuration should indeed be an equilateral triangle, exactly as observed.

What are the symmetries of this spatial configuration. One can see that the only rotations preserving this spatial configuration are rotations by 120 and 240 degrees
around an axis $\alpha$ which is orthogonal to the plane formed by the fish and which passes through the center of the fish triangle.

**What are the optimal motions?** As we have mentioned, in general, a motion can be characterized by a pair $(e, -e)$ of opposite unit vectors. According to our main result, the optimal dynamic configuration – consisting of the fish locations and of the motion-describing pair $(e, -e)$ – must be invariant with respect to all the corresponding symmetries – i.e., in our case, with respect to 120- and 240-degree rotations around $\alpha$.

One can easily check that if the vector $e$ is not parallel to $\alpha$, then the corresponding dynamic configuration is not invariant with respect to such rotations: its orthogonal-to-$\alpha$ component changes when we rotate. Thus, the only invariant direction of motion is in the direction of $\alpha$, i.e., in the direction orthogonal to the fish plane.

And since we proved that the optimal direction should be invariant, we thus conclude that in the three fish case, the motion corresponding to the optimal dynamic configuration is the motion in the direction orthogonal to the fish triangle – which is exactly what was observed.

**Summarizing.** In both cases, symmetry-based approach to optimization shows that the optimal spatial configuration and the optimal direction of motion are exactly what was observed in the recent experiments. Thus, both observed tendencies can be explained by the fact that fish act optimally.

This conclusion does not depend on what exactly optimality criterion the fish uses, as long as it is invariant – as it should be – with respect to all possible rotations, shifts, and permutations of fish.

**Acknowledgments**

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395 (Center for Collective Impact in Earthquake Science C-CIES), and by the AT&T Fellowship in Information Technology.

It was also supported by a grant from the Hungarian National Research, Development and Innovation Office (NRDI).

**References**

Why Two Fish Follow Each Other but Three Fish Form a School