Is Alaska Negative-Tax Arrangement Fair? Almost: Mathematical Analysis

Chon Van Le and Vladik Kreinovich

Abstract In the State of Alaska there is no state income tax. Instead, there is a negative tex: every year every resident gets some money from the state. At present, every resident – from the poorest to the richest – gets the exact same amount of money: in 2024, it is expected to be around \$1500. A natural question is: Is this fair? Maybe poor people should get more since their needs are greater? Maybe the rich people should get proportionally more, since fairness means equal added happiness for all, and for rich people, extra \$1500 is barely noticeable? There have been many ethical discussions about this. In this paper, we analyze the problem from the mathematical viewpoint, and we show that the current arrangement, while not exactly the most fair, is close to the fair one – at least much closer to fairness than the alternative proportional arrangement.

1 Introduction

What is negative tax and how it is arranged. The US state of Alaska is one of the few places in the world where, instead of paying taxes (i.e., paying money to the Government), people receive a "negative tax" – an annual amount of money. At present, the negative tax arrangements are very straightforward: every resident gets the exact same amount of money, irrespective of their other income. A poor person gets the same amount as a millionaire.

But is it fair? A natural question is: is this arrangement fair?

Chon Van Le
International University, Vietnam National University – Ho Chi Minh City
Ho Chi Minh City, Vietnam, e-mail: lvchon@hcmiu.edu.vn

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: vladik@utep.edu

On the one hand, a millionaire does not need extra money, while for a poor person, every dollar counts, so why not give the whole amount only to the poor folks?

On the other hand, if we want to be fair, we may want to make sure that each person gets the same pleasure out of his/her money. To a poor person, receiving \$1500 – this is an estimated 2024 per-person amount – is significant, while for a millionaire it is barely noticeable. So should not we give more to richer people to make it more fair? After all, the usual taxes are proportional to the income, so why should not the negative tax be proportional to the income?

How this issue is usually discussed. As with many other finance-related issues, the issue of fairness of Alaska negative tax is usually discussed on the qualitative ethical level.

What we do in this paper. In this paper, we provide a mathematical analysis of the problem. As a result of this analysis, we show that the current Alaska negative tax arrangement is almost fair. To be more precise, we show – honestly, somewhat contrary to our own intuition – that in the fair arrangement, the amount should slightly increase with income, but increase very slowly – so that the richest person gets twice the amount of the poorest one.

From this viewpoint, the current arrangement when everyone gets the same amount is closer to the optimal distribution than the proportional idea:

- in the actual arrangement, the richest person gets the same amount as the poorest person,
- in the optimal arrangement, the richest person gets twice as much as the poorest person, while
- in the proportional arrangements, the richest person would get thousands of times more than the poorest person.

2 Our analysis

What we mean by fair. The problem of distributing the excess income is a particular case of the general problem of cooperative decision making, when we start with the status quo state, and we compare different alternatives each of which is better, for all participants, than the status quo. Such situations have been analyzed in the 1950s by the (future Nobelist) John Nash [5, 6, 7] in the framework of decision theory (see, e.g., [1, 2, 4, 5, 8, 9, 10]).

According to decision theory, preferences of a rational person can be described by a special function – called *utility function* – that assigns, to each alternative A, a number $u(A)$ such that the person prefers A to B if and only if the utility $u(A)$ is larger than the utility $u(B)$. Utility is usually defined in such a way that if we have an alternative A that leads to outcomes A_i with probabilities p_i, then the utility of A is equal to $u(A) = p_1 \cdot u(A_1) + \ldots + p_n \cdot u(A_n)$.

It is known that these conditions define the utility function modulo an increasing linear transformation, i.e.:

- if the function $u(A)$ correctly describes the person's preferences, then, for each values c_0 and $c_1 > 0$, the function $v(A) \stackrel{\text{def}}{=} c_0 + c_1 \cdot u(A)$ describes the same preferences, and
- if two functions $u(A)$ and $v(A)$ describe the same preferences, then there exist real numbers c_0 and $c_1 > 0$ for which $v(A) = c_0 + c_1 \cdot u(A)$ for all A.

In the cooperative decision making, we have N agents with utility functions $u_1(A)$, ..., $u_N(A)$. Since we have a fixed status quo state A_0, we can replace each original utility function with an equivalent function $U_i(A) \stackrel{\text{def}}{=} u_i(A) - u_i(A_0)$ for which $U_i(A_0) = 0$. With this restriction, the utility functions are still not uniquely determined: for each i and for each value c_i, we can still replace the original utility function $U_i(A)$ with an equivalent re-scaled function $c_i \cdot U_i(A)$ that describes the same preferences.

Based on the values $U_1(A), \ldots, U_N(A)$ corresponding to different alternatives A, we must decide which alternative is better. It makes sense to require that our choice should not depend on renaming the participants. It also makes sense to require that the selection should not change if we replace each utility function $U_i(A)$ with an equivalent one $c_i \cdot U_i(A)$. It also makes sense to require that if for all participants A is better than B, then out of two options A and B the group should select A.

Nash has proven than under these reasonable conditions, the group should select the alternative for which the product of the utilities $U_1(A) \cdot \ldots \cdot U_N(A)$ is the largest possible. This is known as *Nash's bargaining solution*. This is what we will use to describe a fair solution.

Let us apply Nash's bargaining solution to our problem. To apply Nash's bargaining solution to our problem, we need to recall how utility depends on money. This is *not* a linear dependence: as we have mentioned earlier, an extra $1500 means a lot to a poor person and practically nothing to a millionaire. Empirical analysis shows that the utility is proportional to the square root of the amount of money x: $u(x) = k \cdot \sqrt{x}$, for some coefficient $k > 0$; see, e.g., [3].

Let v_i denote the original income of the i-th person – before the negative tax. This means that at the status quo state, the i-th person has utility $u_i(A_0) = k_i \cdot \sqrt{v_i}$, for some coefficient k_i. If we give an additional amount t_i to the i-th person, then his/her utility becomes equal to $u_i(A) = k_i \cdot \sqrt{v_i + t_i}$. So, the re-scaled utility value – for which the utility of the status quo alternative is 0 – is equal to

$$U_i = u_i(A_i) - u_i(A_0) = k_i \cdot \sqrt{v_i + t_i} - k_i \cdot \sqrt{v_i} = k_i \cdot \left(\sqrt{v_i + t_i} - \sqrt{v_i}\right).$$

So, if we denote the overall amount of the money to be distributed by $T = t_1 + \ldots + t_N$, then the Nash's bargaining solution takes the following form.

Mathematical formulation of the problem. We are given the value $T > 0$ and the non-negative values $v_1 \geq 0, \ldots, v_N \geq 0$. Among all the tuples $t_1 \geq 0, \ldots, t_N \geq 0$ that satisfy the constraint

$$t_1 + \ldots + t_N = T, \tag{1}$$

we must find the tuple for which the product

$$k_1 \cdot (\sqrt{v_1 + t_1} - \sqrt{v_1}) \cdot \ldots \cdot k_N \cdot \left(\sqrt{v_N + t_N} - \sqrt{v_N}\right) \quad (2)$$

is the largest possible.

Let us make the problem somewhat simpler. To make the problem somewhat simpler, let us first notice that if $a > b$ and we multiply both values by the same positive constant k, we still have $k \cdot a > k \cdot b$. Similarly, inequalities do not change if we divide both sides by the same positive number. Thus, if we divide all the values of the objective function (2) by a positive number $k_1 \cdot \ldots \cdot k_N$, this will not change which tuples have a larger value of this function and which have smaller value. Thus, instead of maximizing the product (2), we can maximize a simpler expression

$$\left(\sqrt{v_1 + t_1} - \sqrt{v_1}\right) \cdot \ldots \cdot \left(\sqrt{v_N + t_N} - \sqrt{v_N}\right). \quad (3)$$

The objective function (3) is a product. From the computational viewpoint, a product is somewhat more complex than a sum. It is known how to reduce a product to a sum – this is what logarithms were invented for. The function $\ln(x)$ is strictly increasing, so maximizing the objective function (3) is equivalent to maximizing its logarithm. Since the logarithm of the product is equal to the product of logarithms, we get the following equivalent problem: under constraint (1), maximize the following expression:

$$\ln\left(\sqrt{v_1 + t_1} - \sqrt{v_1}\right) + \ldots + \ln\left(\sqrt{v_N + t_N} - \sqrt{v_N}\right). \quad (4)$$

Let us solve the problem. To solve the constraint optimization problem, we can use the usual Lagrange multiplier method and thus, reduce it to the following unconstrained optimization problem: maximize the expression

$$\ln\left(\sqrt{v_1 + t_1} - \sqrt{v_1}\right) + \ldots + \ln\left(\sqrt{v_N + t_N} - \sqrt{v_N}\right) + \lambda \cdot (t_1 + \ldots + t_N - T), \quad (5)$$

for some coefficient λ.

To find the minimum of the expression (5), we can simply differentiate this expression with respect to each unknown t_i and equate the resulting derivative to 0. As a result, we get the following equality:

$$\frac{1}{\sqrt{v_i + t_i^{\text{opt}}} - \sqrt{v_i}} \cdot \frac{1}{2 \cdot \sqrt{v_i + t_i^{\text{opt}}}} + \lambda = 0. \quad (6)$$

If we multiply the two fractions by multiplying their numerators and denominators and take into account that the product of two square roots is the original value, we conclude that

$$\frac{1}{2 \cdot \left(v_i + t_i^{\text{opt}} - \sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)}\right)} + \lambda = 0. \quad (7)$$

If we multiply both sides of this equality by 2 and move the resulting term 2λ to the right-hand side, we get

$$\frac{1}{v_i + t_i^{\text{opt}} - \sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)}} = -2\lambda. \tag{8}$$

If we now take an inverse of both sides, we get

$$v_i + t_i^{\text{opt}} - \sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)} = t_0, \tag{9}$$

where we denoted

$$t_0 \stackrel{\text{def}}{=} -\frac{1}{2\lambda}.$$

What will happen in extreme cases? Before we consider the general case, let us analyze what will happen in the two extreme vases: of the poorest person for whom $v_i = 0$ and of the richest person for whom $v_i \to \infty$. For the poorest person case, when $v_i = 0$, the equation (9) leads to $t_i^{\text{opt}} = t_0$.

For the richest person case when $v_i \to \infty$, we have

$$\sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)} = \sqrt{v_i^2 \cdot \left(1 + \frac{t_i^{\text{opt}}}{v_i}\right)} = v_i \cdot \sqrt{1 + \frac{t_i^{\text{opt}}}{v_i}}.$$

The value t_i is bounded by T while v_i tends to infinity. Thus, the ratio t_i^{opt}/v_i tends to 0. In general,

$$\sqrt{1+\varepsilon} = 1 + \frac{1}{2} \cdot \varepsilon + O(\varepsilon^2).$$

Thus, we get

$$\sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)} = v_i \cdot \left(1 + \frac{t_i^{\text{opt}}}{2v_i} + O\left(\left(\frac{t_i^{\text{opt}}}{2v_i}\right)^2\right)\right) = v_i + \frac{t_i^{\text{opt}}}{2} + o(1).$$

Thus, in the limit $v_i \to \infty$, the equation (9) takes the form

$$v_i + t_i^{\text{opt}} - v_i - \frac{t_i^{\text{opt}}}{2} = t_0,$$

i.e.,

$$\frac{t_i^{\text{opt}}}{2} = t_0$$

and $t_i^{\text{opt}} = 2t_0$. So, in the Nash's fair arrangement, the richest person indeed gets twice as much as the poorest person.

What happens in the general case. The general case can be described by the following proposition:

Proposition. *The solution $t_i^{\rm opt}$ to the equation* (9) *is always between t_0 and $2t_0$.*

Comment. For readers' convenience, the proof is place in a special (last) section.

How can we actually compute the fair solution? As we show in the proofs section, once we know t_0, we can explicitly compute all the values $t_i^{\rm opt}$ by using a straightforward formula

$$t_i^{\rm opt}=t_0-\frac{v_i}{2}+\sqrt{\frac{v_i^2}{4}+v_i\cdot t_0}. \tag{10}$$

The value t_0 can then be found if we substitute the expressions (10) into the formula (1) and thus, get the following equation with one unknown (which is, thus, easy to solve):

$$\sum_{i=1}^N\left(t_0-\frac{v_i}{2}+\sqrt{\frac{v_i^2}{4}+v_i\cdot t_0}\right)=T.$$

Comment. Let us show that the formula (10) agrees with both extreme cases mentioned above. Indeed, for $v_i=0$, we clearly have $t_i^{\rm opt}=t_0$. For $v_i\to\infty$, we have

$$\sqrt{\frac{v_i^2}{4}+v_i\cdot t_0}=\sqrt{\frac{v_i^2}{4}\cdot\left(1+\frac{4t_0}{v_i}\right)}=\frac{v_i}{2}\cdot\left(1+\frac{2t_0}{v_i}+o\right)=\frac{v_i}{2}+t_0+o(1).$$

Thus, the expression (10) takes the following form:

$$t_i^{\rm opt}=t_0-\frac{v_i}{2}+\frac{v_i}{2}+t_0=o(1)=2t_0+o(1),$$

i.e., in the limit, we indeed get $t_i^{\rm opt}=2t_0$.

3 Proofs

Proof of the Proposition. For $v_i=0$, the equation (9) leads to $t_i^{\rm opt}=t_0$. Thus, to prove the Proposition, it is sufficient to consider the case when $v_i>0$.

1°. Let us first prove, by contradiction, that we cannot have $t_i^{\rm opt}=0$ for some i.

Indeed, in this case, the corresponding utility U_i is 0, so the product of utilities is 0. Thus, it cannot be the largest possible value, since if we simply divide $T>0$ into N equal parts, we get all utilities positive – and thus, the positive product of utilities.

2°. Let us now prove that $t_0>0$.

Due to equation (9), the desired inequality is equivalent to

$$v_i + t_i^{\rm opt} - \sqrt{v_i \cdot \left(v_i + t_i^{\rm opt}\right)} > 0,$$

i.e., to

$$v_i + t_i^{\rm opt} > \sqrt{v_i \cdot \left(v_i + t_i^{\rm opt}\right)}. \quad (11)$$

Both sides are non-negative. For non-negative numbers, the function $x \mapsto x^2$ is strictly increasing. So, the inequality (11) is equivalent to what we will get by squaring both sides:

$$v_i^2 + 2v_i \cdot t_i^{\rm opt} + \left(t_i^{\rm opt}\right)^2 > v_i^2 + v_i \cdot t_i^{\rm opt}.$$

Subtracting the right-hand side from the left-hand side, we get the equivalent inequality $v_i \cdot t_i^{\rm opt} + \left(t_i^{\rm opt}\right)^2 > 0$. This inequality is clearly true, since $v_i \ge 0$ and $t_i^{\rm opt} > 0$. Thus, the original inequality $t_0 > 0$ is also true. The statement is proven.

3°. In the equation (9), if we move t_0 to the left-hand side, we get an equivalent equation $L\left(t_i^{\rm opt}\right) = 0$, where we denoted

$$L(t_i) \stackrel{\rm def}{=} v_i + t_i - \sqrt{v_i \cdot (v_i + t_i)} - t_0. \quad (12)$$

4°. Let us prove that the expression $L(t_i)$ is a strictly increasing function of t_i.

For this purpose, it is sufficient to prove that the partial derivative of $L(t_i)$ with respect to t_i is always positive. Indeed, by differentiating the expression (11) with respect to t_i, we conclude that

$$\frac{\partial L}{\partial t_i} = 1 - \frac{v_i}{2\sqrt{v_i \cdot (v_i + t_i)}}.$$

Here, $v_i \cdot (v_i + t_i) \ge v_i^2$, thus $v_i \le \sqrt{v_i \cdot (v_i + t_i)}$. Thus,

$$\frac{v_i}{2\sqrt{v_i \cdot (v_i + t_i)}} \le \frac{1}{2}$$

and

$$\frac{\partial L}{\partial t_i} \ge 1 - \frac{1}{2} = \frac{1}{2} > 0.$$

The statement is proven.

5°. Let us now prove that for $t_i = t_0$, we have $L(t_0) < 0$ (remember that we assumed that $v_i > 0$).

Indeed, the desired inequality has the form

$$v_i + t_0 - \sqrt{v_i \cdot (v_i + t_0)} - t_0 = v_i - \sqrt{v_i \cdot (v_i + t_0)} < 0,$$

i.e., equivalently, the form $v_i < \sqrt{v_i \cdot (v_i + t_0)}$. Here, both sides are non-negative, so we can get an equivalent inequality by squaring both sides:

$$v_i^2 < v_i \cdot (v_i + t_0) = v_i^2 + v_i \cdot t_0.$$

By subtracting v_i^2 from both sides, we get $0 < v_i \cdot t_0$. This is clearly true since $v_i > 0$ and $t_0 > 0$. Thus, the equivalent inequality $L < 0$ is true too.

6°. Let us now prove that for $t_i = 2t_0$, we have $L(2t_0) > 0$.

Indeed, the desired inequality has the form

$$v_i + 2t_0 - \sqrt{v_i \cdot (v_i + 2t_0)} - t_0 = v_i + t_0 - \sqrt{v_i \cdot (v_i + 2t_0)} > 0,$$

i.e., equivalently, the form $v_i + t_0 > \sqrt{v_i \cdot (v_i + 2t_0)}$. Here, both sides are non-negative, so we can get an equivalent inequality by squaring both sides:

$$v_i^2 + 2v_i \cdot t_0 + t_0^2 > v_i \cdot (v_i + t_0) = v_i^2 + v_i \cdot t_0.$$

By subtracting $v_i^2 + 2v_i \cdot t_0$ from both sides, we get $t_0^2 > 0$. This is clearly true since $t_0 > 0$. Thus, the equivalent inequality $L > 0$ is true too.

7°. Now, we are ready to prove the proposition.

Indeed, according to Part 4 of this proof, the function $L(t_i)$ is strictly increasing. It is negative for $t_i = t_0$, it is positive for $t_i = 2t_0$, so the value t_i^{opt} for which $L\left(t_i^{\text{opt}}\right) = 0$ must indeed be between t_0 and $2t_0$.

The Proposition is proven.

Proof of the formula (10). If we move the square root to the right-hand side of the formula (9) and t_0 to the left-hand side, we get the following formula:

$$v_i + t_i^{\text{opt}} - t_0 = \sqrt{v_i \cdot \left(v_i + t_i^{\text{opt}}\right)}. \tag{13}$$

Squaring both sides and opening the parentheses in the right-hand side, we get

$$v_i^2 + 2v_i \cdot t_i^{\text{opt}} - 2v_i \cdot t_0 + \left(t_i^{\text{opt}}\right)^2 - 2t_i^{\text{opt}} \cdot t_0 + t_0^2 = v_i^2 + v_i \cdot t_i^{\text{opt}}. \tag{14}$$

Subtracting v_i^2 from both sides, moving all the terms to the left-hand side, and combining terms proportional to t_i^{opt}, we get the following quadratic equation for determining t_i^{opt}:

$$\left(t_i^{\text{opt}}\right)^2 + t_i^{\text{opt}} \cdot (v_i - 2t_0) + (t_0^2 - 2v_i \cdot t_0) = 0. \tag{15}$$

For an equation $x^2 + p \cdot x + q = 0$, the general solution is

$$x = -\frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}.$$

For the equation (15), this leads to

$$t_i^{\rm opt} = t_0 - \frac{v_i}{2} \pm \sqrt{\left(t_0 - \frac{v_i}{2}\right)^2 - t_0^2 + 2v_i \cdot t_0}. \tag{16}$$

The expression under the square root is equal to

$$t_0^2 - v_i \cdot t_0 + \frac{v_i^2}{4} - t_0^2 + 2v_i \cdot t_0.$$

The terms proportional to t_0^2 cancel each other, and the terms $-v_i \cdot t_0$ and $2v_i \cdot t_0$ lead to $v_i \cdot t_0$. Thus, the expression under the square root is equal to

$$\frac{v_i^2}{4} + v_i \cdot t_0,$$

and the formula (16) takes the form

$$t_i^{\rm opt} = t_0 - \frac{v_i}{2} \pm \sqrt{\frac{v_i^2}{4} + v_i \cdot t_0}. \tag{17}$$

We have proven, in the proof of the Proposition, that $t_i^{\rm opt}$ is always greater than or equal to t_0. Thus, we cannot have the minus sign in this formula. So, we must have plus. Hence, we get the desired formula (10).

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