Fair Economic Division: How to Modify Shapley Value to Take Into Account that Different People Have Different Productivity

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Abstract

Purpose: When several participants, working together, gained some amount of money, what is the fair way to distribute this amount between them? This is the problem that the future Nobelist Lloyd Shapley was working on when he proposed what is now called the Shapley value – a division uniquely determined by natural fairness assumptions. However, this solutions is not universal: it assumes that all participants are equal – in particular, that they have equal productivity. In practice, people have different productivity levels, and these productivity levels can differ a lot: e.g., some software engineers are several times more productive than others. It is desirable to take this difference in productivity into account.

Design/methodology/approach: Shapley value is based on an axiomatic approach: it is uniquely determined by the appropriate fairness assumptions. To generalize Shapley value to the case of different productivity, we modified these assumptions appropriately, and analyzed what can be derived from these modified assumptions.

Findings: We prove that there is a unique division scheme that satisfies all the resulting assumptions. This scheme is thus a generalization of Shapley value to this more general and more realistic situation, when different participants have different productivity.

Originality/value: Both the formulation of the problem and the result are new. The resulting division scheme can be used to more adequately distribute the common gains – by explicitly taking into account that different participants have, in general, different productivity.

Keywords: Shapley value, Fair distribution, Difference in productivity, Axiomatic approach.

1 Formulation of the problem

Fair division: a problem. Let us assume that a group $N_n \stackrel{\text{def}}{=} \{1, \ldots, n\}$ of n people jointly gets some benefit $v(N_n)$. What is the fair way to distribute this benefit between the participants, i.e., to assign values $\varphi_1, \ldots, \varphi_n$ whose sum is $v(N_n)$?

What do we need to know to make a fair division. To make a fair distribution, it is important to know what is the contribution of each participant. This can be described by providing, for each subset $S \subseteq N$, a value $v(S)$ that people from S would have gained if they acted on their own, without help of others. So, we get a function $S \mapsto v(S)$ that characterizes the situation.

Shapley value. A Nobelist Lloyd Shapley found out that under reasonable conditions, there is only one way $\varphi_i(v)$ to assign the distribution to each such function $v(S)$; see, e.g., (Shapley, 1951), (Shapley, 1953), (Roth, 1988), (Luce and Raiffa, 1989), (Myerson, 1997), and (Owen, 2013). This distribution has many equivalent forms, In this paper, we use the following form

$$\varphi_i(v) = \sum_{S: i \in S} \frac{t(S)}{|S|}, \tag{1}$$

where $|S|$ denotes the number of elements in the set S and

$$t(S) \stackrel{\text{def}}{=} \sum_{R \subseteq S} (-1)^{|S|-|R|} \cdot v(R). \quad (2)$$

What are the requirements behind Shapley value. Shapley's first condition is *symmetry*: if two participants i and j contribute equally, i.e., if the values $v(S)$ do not change when we swap i and j, then these participants should get equal amounts: $\varphi_i(v) = \varphi_j(v)$.

Shapley's second condition is that if a person i is not contributing, i.e., if $v(S \cup \{i\}) = v(S)$ for all S, then we should have $\varphi_i(v) = 0$.

Shapley's third condition is additivity: if we have two situations $u(S)$ and $v(S)$, then we can:

- either consider them separately
- or view them as a single situation with gain $w(S) = u(S) + v(S)$.

The outcome should not depend on how we view this, so we should have $\varphi_i(w) = \varphi_i(u) + \varphi_i(v)$.

Comment. In this paper, we will only deal with economic applications of Shapley value. It should be mentioned that Shapley value is now also actively used in machine learning, to find the importance $\varphi_i(v)$ of each of n features based on the effectiveness $v(S)$ of solving the problem when we only use features from the set S.

Why go beyond Shapley value. Symmetry makes perfect sense if all participants are equally productive. In reality, people have different productivity: e.g., some programmers are several times more productive than others. If we naively apply Shapley value to compute each person's bonus, more productive participants will get the exact same amount as less productive ones, which is not fair. It is therefore desirable to take productivity p_i of each participant into account. In other words, we need to determine the values φ_i based on both $v(S)$ and the values $p = (p_1, p_2, \ldots)$: $\varphi_i(v, p)$.

What we do in this paper. In this paper, we show how to adjust the requirement behind the Shapley value so that they would lead to a unique determination of the desired distributions $\varphi_i(v,p)$.

2 What we propose

Natural first requirement. If i's productivity is twice larger than j's, this means that the company can replace i with two less productive workers and get the same result. After this replacement, all participants have the same productivity, so to this replaced situation, we can apply symmetry and get Shapley value – and then assign to i the sum of bonuses that Shapley value recommends for his/her two replacements.

Similarly, if we replace person i with 3 or more workers, it makes sense to require that the amount given to the person i should be equal to the sum of amounts given to these workers. It turns out that it is sufficient to require this property only for situation which are]it simple – in some precise sense described below.

Natural second requirement. If the productivity changes a little bit, the resulting distribution should also change a little bit. In mathematical terms, this means that the dependence of distribution on productivity should be continuous.

Main result. If impose these two additional requirements, then we get the following result.

Definition 1. *By a* situation*, we mean a triple* (n,v,p)*, where:*

- n *is a positive integer,*
- v *is a function that assigns a value* $v(S)\ge 0$ *to each subset* $S\subseteq N_n \stackrel{\text{def}}{=} \{1,\ldots,n\}$ *and for which* $R\subseteq S$ *implies* $v(R)\le v(S)$*, and*
- $p=(p_1,\ldots,p_n)$ *is a tuple of* n *positive numbers.*

Definition 2. *We say that the situation* (n,v,p) *is* simple *if for some subset*

$R \subseteq N_n$, we have $v(S) = 0$ if $R \not\subseteq S$ and $v(S) = v(R)$ otherwise. We will call this set R basic.

Definition 3.

- *By a* division strategy, *we mean a function $\varphi(n, v, p)$ that assigns, to each situation (n, v, p), an n-tuple of real numbers $\varphi_i(n, v, p)$, $1 \le i \le n$, for which $\varphi_1(n, v, p) + \ldots + \varphi_n(n, v, p) = v(N_n)$.*

- *We say that a division strategy is* symmetric *if for every situation in which swapping i and j does not change v and p, i.e., in which $p_i = p_j$ and $v(S \cup \{i\}) = v(S \cup \{j\})$ for all sets S that contain neither i nor j, we have $\varphi_i(n, v, p) = \varphi_j(n, v, p)$.*

- *We say that a division strategy has the* null-player property *if for each situation and for each player i for which $v(S \cup \{i\}) = v(S)$ for all S, we get $\varphi_i(n, v, p) = 0$.*

- *We say that a division strategy is* additive *if for all u and v, we have $\varphi_i(n, u + v, p) = \varphi_i(n, u, p) + \varphi_i(n, v, p)$ for all i.*

- *We say that a division strategy is* contibuous *if $\varphi(n, v, p)$ ia a continuous function of p.*

- *Wesa that a division strategy is* productivity-based *if for every simple situation with a basic set R, if we combine participants from a subset $R' \subseteq R$ into a single participant i_0 with productivity equal to the sum of productivities of all members of R', then in this new situation (n', v', p'),*

$$\varphi_{i_0}(n', v', p') = \sum_{i \in R'} v_i(n, n, p).$$

Proposition. *There exists one and only one division strategy which is symmetric, has null-player property, is additive, continuous, and productivity-based. In this strategy,*

$$\varphi(n, v, p) = p_i \cdot \sum_{S: i \in S} \frac{t(S)}{p(S)}, \tag{3}$$

where we denoted

$$p(S) \stackrel{\text{def}}{=} \sum_{i \in S} p_i. \tag{4}$$

Comment. One can easily see that if all participants have the same productivity, i.e., if $p_1 = \ldots = p_n$, then the formula (3) becomes the usual Shapley value formula (1).

3 Proof

1°. It is relatively easy to prove that the strategy (3) satisfied all the above properties. So, to complete the proof, it is sufficient to prove that if a division strategy satisfies all the above properties, then it has the form (3).

2°. Let us assume that a division strategy satisfies all the above properties. Let us first show that for each simple situation (n, v, p) with a basic set R:

- we have $\varphi_i(n, v, p) = 0$ for all $i \notin R$ and
- we have

$$\varphi_i(n, v, p) = p_i \cdot \frac{v(R)}{p(R)}. \tag{5}$$

Let us prove these two formulas one by one.

2.1°. From the null-player property, it follows that $\varphi_i(n, v, p) = 0$ for all $i \notin R$ – since these participants do not add anything to any value $v(S)$.

2.2°. Let us first prove formula (5) for the cases when all theb productivity values are rational numbers, i.e., ratios of natural numbers. In this case, we can bring all these rational numbers to a common denomination d, so we have

$$p_i = \frac{n_i}{d}$$

for some integers n_i. In this case, we can replace each participant i with n_i participants with productivity $1/d$. After this substitution, we get a new situation with $\sum n_j$ participants with the same productivity. So, due to symmetry, each of them will get the exact same share, i.e., the value

$$\frac{v(N)}{\sum\limits_j n_j} = \frac{v(R)}{\sum\limits_j n_j}.$$

Due to the fact the division is productivity-based, each participants i in the original situation gets n_i times more, i.e., gets the value

$$\varphi(n, v, p) = n_i \cdot \frac{v(R)}{\sum\limits_j n_j}.$$

We can divide both the numerator and the denominator of this expression by d, this will not change the resulting values. So, we get

$$\varphi(n, v, p) = \frac{n_i}{d} \cdot \frac{v(R)}{\sum\limits_j \frac{n_j}{d}},$$

i.e.,

$$\varphi(n, v, p) = p_i \cdot \frac{v(R)}{\sum\limits_j p_j}.$$

By definition of $p(R)$, this means that

$$\varphi(n, v, p) = p_i \cdot \frac{v(R)}{p(R)}.$$

We have proved the formula (5) for rational productivity values p_i. Any real-valued productivities can be approximated, with any given accuracy, by rational numbers. Thus, in the limit when these rational approximations tend to the original values p_i, we conclude – dut to continuity – that the formula (5) holds for real values as well.

3°. It is known that for every function v, we have

$$v(S) = \sum_{R \subseteq S} t(R).$$

Some of the value $t(R)$ may be negative. If we move these values to the left-hand side, we get the following equality

$$v(S) + \sum_{R \subseteq S: t(R) < 0} |t(R)| = \sum_{R \subseteq S: t(R) \geq 0} t(R).$$

For every set R, let us consider a simple situation v_R with this basic set R and $v_R(R) = |t(R)|$, then we have

$$v(S) + \sum_{R \subseteq S: t(R) < 0} v_R(S) = \sum_{R \subseteq S: t(R) \geq 0} v_R(S).$$

Thus, by additivity,

$$\varphi_i(n, v, p) + \sum_{R: i \in R \,\&\, t(R) < 0} \varphi_i(n, v_R, p) = \sum_{R: i \in R \,\&\, t(R) \geq 0} \varphi_i(n, v_R, p).$$

From Part 2 of this proof, we know the values $\varphi(n, v_R.p)$ for all simple situations, Thus, we get

$$\varphi_i(n, v, p) + \sum_{R: i \in R \,\&\, t(R) < 0} p_i \cdot \frac{|t(R)|}{p(R)} = \sum_{R: i \in R \,\&\, t(R) \geq 0} p_i \cdot \frac{t(R)}{p(R)}.$$

If we move the sum from the left-hand side into the right-hand side, we get the formula

$$\varphi_i(n, v, p) = \sum_{R: i \in R} p_i \cdot \frac{t(R)}{p(R)}.$$

If we have the common factor out of the sum, we get the desired formula (3).

The proposition is proven.

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