How to Share a Success, How to Share a Crisis, and How All This Is Related to Fuzzy

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Abstract. In many practical situations, a group of people needs to share a success. What is the fair way to share this success? Nobelist John Nash showed that under reasonable conditions, the group should select the alternative for which the product of utility gains is the largest possible. This solution makes perfect sense from the fuzzy-formalized commonsense viewpoint: it maximizes the degree of confidence that all participants are happy. A natural question is: can we extend this result to a different class of situations, when a group of people needs to share sacrifices caused by a crisis? In this paper, we prove that in this case, no solution satisfies the same set of conditions. We also explain how to actually fairly distribute needed sacrifices in the case of a crisis.

Keywords: fair division · fair distribution of sacrifices · Nash's bargaining solution · fuzzy techniques

1 Formulation of the Problem

How to share a success: a problem. Often, a group of people has an opportunity to benefit all its members. For example, family members get an inheritance in which their late relative did not specify who gets what. In many such cases, there are many possible alternative decisions. In some of these possible alternatives, some of the participants benefit more, in others, other participants benefit more. Which of these alternatives should we choose?

Known solution: Nash's bargaining solution and its relation to fuzzy. This problem is well studied in decision theory; see, e.g., [2, 3, 5, 6, 10, 11, 14]. The solution to this problem was provided by John Nash – who later won a Nobel prize for his research; see, e.g., [6, 8, 9]. He showed that under some reasonable requirements (that we will describe later), the group should select an alternative for which the product $U_1 \cdot U_2 \cdot \ldots \cdot U_n$ of their utility gains U_i is the largest possible. This solution is known as *Nash's bargaining solution.*

This solution has a natural interpretation in fuzzy logic; see, e.g., [1, 4, 7, 12, 13, 15]. Indeed, it is reasonable to make sure that everyone is as happy as possible, i.e., that the 1st person is happy *and* the 2nd person is happy, etc. Thus, it makes sense to select an alternative for which our degree of confidence in the statement "the 1st person is happy and the 2nd person is happy, etc." is the largest possible. It is natural to use utility gain U_i as the measure of happiness. To combine these degrees, we can use one of the most widely used "and"-operations: algebraic product. Then, our degree of confidence that an alternative makes everyone happy is equal to the product of utilities – which is exactly what Nash's bargaining solution is about.

But what if there is a crisis? Nash's bargaining solution is only applicable in situations of success, when everyone gains. But sometimes, we encounter the opposite situation – of a crisis, when everyone needs to sacrifice. What is the fair way to share a crisis?

What we do in this paper. In this paper, we analyze what is the fair way to share a crisis, i.e., a situation when we need to decrease utilities.

Since our starting point is utility-based Nash's bargaining solution, we start the paper with Section 2 that reminds the reader what is utility and how Nash's bargaining solution is justified. In Section 3, we show that a similar approach – based on natural requirements – does not work for the case of a crisis. In Section 4, we provide a different set of requirements and show that it enables us to come up with an (almost) unique fair solution. For readers' convenience, all the proofs are placed in special Section 5.

2 Utility and Nash's Bargaining Solution: A Brief Reminder

What is utility. One of the main objectives of decision theory is to help people make decisions in complex situations. In situations in which there is a very large number of alternatives, it is not possible for a person to process all this data by hand, we need to use computers.

Information about different alternatives comes in different formats, with words, etc. Computers, however, are not very good in processing words, they are much better in processing numbers – this is what they were originally designed for. So, to effectively use computers, we need to describe all the available information in numerical terms. In particular, we need to describe people's preferences in numerical form. For this description, the notion of utility was invented.

This notion allows us to assign, to each alternative a, a number $u(a)$ – called its *utility* – so that a is preferable to b if and only if $u(a)$ is larger than $u(b)$. To describe the utilities, we need to select two extreme alternatives, ideally not realistic:

– a very bad alternative a_- which is worse than any actual alternatives, and
– a very good alternative a_+ which is better than any actual alternative.

Once these alternatives are selected, we can form, for each value p from the interval $[0, 1]$, a *lottery* $L(p)$ in which we get a_+ with probability p and a_- with the remaining probability $1-p$. Of course, the larger the probability p of getting a very good outcome, the better the lottery: if $p > p'$ then, for the user, the lottery $L(p)$ is better than the lottery $L(p')$; we will denote this preference by $L(p) \succ L(p')$.

To find a numerical value $u(a)$ corresponding to an alternative a, we need to compare a with lotteries $L(p)$ corresponding to different values $p \in [0,1]$. For each p:

- either a is better ($a \succ L(p)$),
- or the lottery is better ($L(p) \succ a$),
- or the alternative has the same value to the user as the lottery; we will denote this by $a \sim L(p)$.

When p is small, the lottery $L(p)$ is close to the very bad alternative a_- and is, therefore, worse than a: $a \succ L(p)$. On the other hand, when p is close to 1, the lottery $L(p)$ is close to the very good alternative a_+ and is, therefore, better than a: $L(p) \succ a$. One can show that there exists a threshold value $u(a)$ that separates the values p for which $a \succ L(p)$ from the values p for which $L(p) \succ a$: this value is equal to

$$u(a) = \sup\{p : a \succ L(p)\} = \inf\{p : L(p) \succ a\}.$$

This threshold value is called the utility of the alternative a.

It can proven that for any set of alternatives $a_1, \ldots, a_n$, if we consider a lottery in which we get a_i with probability p_i, then the utility of this lottery is equal to $p_1 \cdot u(a_1) + \ldots + p_n \cdot u(a_n)$.

Comment. At first glance, this notion may appear to be not very practical: there are infinitely many possible values p, so comparing the alternative a with all these lotteries may take forever. However, this is not an obstacle: we can find the utility value really fast if we use bisection. Namely, we start with the interval $[\underline{u}, \overline{u}] = [0,1]$ that contains the actual (unknown) value $u(a)$. At each iteration, we decrease this interval – while making sure that the shrank interval still contains $u(a)$. Namely, we compute the midpoint $\widetilde{u}$ of the current interval, and compare the alternative a with the lottery $L(\widetilde{u})$.

- If a is better than $L(\widetilde{u})$, this means that $u(a)$ is located in the interval $[\widetilde{u}, \overline{u}]$.
- If a is worse than $L(\widetilde{u})$, this means that $u(a)$ is located in the interval $[\underline{u}, \widetilde{u}]$.

On each iteration, we make one comparison, and the interval becomes twice narrower. In k iterations, we thus get the interval of width 2^{-k}. We stop when the width of this interval because smaller than a given accuracy ε. This way, the midpoint of the resulting interval approximates $u(a)$ with accuracy $\varepsilon/2$. So, in 6 iteration, we reach accuracy 1%, and in 9 iterations, we reach accuracy 0.1%.

Utility is defined modulo a strictly increasing linear transformation. The above definition of utility depends on the selection of the two extreme alternatives a_- and a_+. If we select a different pair (a'_-, a'_+), then, in general, we get different numerical values of the utility. It can be proven that the new utility values $u'(a)$ can be obtained from the original ones $u(a)$ by a strictly increasing linear transformation. In precise terms, there exist constants $c > 0$ and d for which, for every alternative a, we have $u'(a) = c \cdot u(a) + d$.

Nash's bargaining solution: natural requirements and the resulting criterion. In the success situations, we start with some starting state, which is known as the *status quo* state, in which the participants' utilities form a tuple $s \stackrel{\text{def}}{=} (s_1, \ldots, s_n)$. We have the set S of different possible alternative in which everyone gains, i.e., in which for the resulting utilities $u = (u_1, \ldots, u_n)$ we have $u_i > s_i$ for all i. For each two possible alternatives u and u', it is also possible, for each value $p \in [0, 1]$, to have a lottery in which we get u with probability p and u' with the remaining probability $1 - p$. As we have mentioned, the utility of this lottery is equal to $p \cdot u + (1 - p) \cdot u'$. This is an example of a convex combination of the two vectors u and u'. Thus, the set S should contain, with every two vectors, its convex combination – i.e., S should be a convex set. We need to come up with a group-based preference relation $\succ_s$ between the tuples.

Let us list natural requirements. First is what is called *Pareto optimality*: if $u_i > u'_i$ for all i (we will denote it by $u > u'$), then we should have $u \succ_s u'$. Second, the preference relation should only depend on the preferences, it should not depend on the choice of a_- and a_+ for each person. In other words, for every two tuples $c = (c_1, \ldots, c_n)$ with $c_i > 0$ for all i and $d = (d_1, \ldots, d_n)$, if we denote

$$T_{c,d}(u) \stackrel{\text{def}}{=} (c_1 \cdot u_1 + d_1, \ldots, c_n \cdot u_n + d_n),$$

then $u \succ_s u'$ should imply $T_{c,d}(u) \succ_{T_{c,d}(s)} T_{c,d}(u')$. Third, preferences should not depend on how we number the participants. If we swap i and j – we will denote this transformation by $\pi_{i,j}$ – then preference should not change, i.e., $u \succ_s u'$, then $\pi_{i,j}(u) \succ_{\pi_{i,j}(s)} \pi_{i,j}(u')$.

Finally, the solution should be fair: equal participants should get equal benefit. In precise terms, if both the status quo state s and the set S do not change under the swap, i.e., if $\pi_{i,j}(s) = s$ and $\pi_{i,j}(S) = S$, then for every vector $u \in S$ there should exist a vector u' which is either better or of the same quality as u (we will denote it by $u' \succeq_s u$) for which $\pi_{i,j}(u') = u'$.

Let us describe these conditions in precise terms.

Definition 1. *Let $n > 1$ be an integer. A binary relation $\succeq$ on a set A is called a* total pre-order *if it satisfied the following three conditions for all a, b, and c:*

- *if $a \succeq b$ and $b \succeq c$, then $a \succeq c$ (*transitivity*),*
- *$a \succeq a$ (*reflexivity*), and*
- *$a \succeq b$ or $b \succeq a$ (*totality*).*

Notations. For each such relation:

- if $a \succeq b$ and $b \not\succeq a$, we will denote it by $a \succ b$, and
- if $a \succeq b$ and $b \succeq a$, we will denote it by $a \sim b$.

Definition 2. *Let $n > 1$ be an integer. We say that we have a* preference relation *if for every vector $s \in \mathbb{R}^n$, we have total pre-order relation $\preceq_s$ on the set of all tuples x for which $x > s$. We say that a preference relation is:*

- Pareto-optimal *if $u > u$ implies $u \succ u'$;*
- scale-invariant *if for every two tuples $c = (c_1, \ldots, c_n)$ with $c_i > 0$ for all i and $d = (d_1, \ldots, d_n)$, $u \succeq_s u'$ implies $T_{c,d}(u) \succeq_{T_{c,d}(s)} T_{c,d}(u')$, where we denoted $T_{c,d}(u) \stackrel{\text{def}}{=} (c_1 \cdot u_1 + d_1, \ldots, c_n \cdot u_n + d_n)$;*
- anonymous *if for every i and j, $u \succeq_s u'$ implies $\pi_{i,j}(u) \succeq_{\pi_{i,j}(s)} \pi_{i,j}(u')$, where $\pi_{i,j}$ swaps elements u_i and u_j in a vector; and*
- fair *if for every vector s and for every convex set S all of whose elements x satisfy the condition $x > s$, once $\pi_{i,j}(s) = s$ and $\pi_{i,j}(S) = S$, then for every vector $u \in S$ there should exist a vector u' which is either better or of the same quality as u (we will denote it by $u' \succeq_s u$) for which $\pi_{i,j}(u') = u'$.*

Proposition 1. *For every n, for every preference relation $\succeq_s$, the following two conditions are equivalent:*

- *the preference relation is Pareto-optimal, scale-invariant, anonymous, and fair;*
- *the preference relation has Nash's form*

$$u \succeq_s u' \leftrightarrow \prod_{i=1}^{n} U_i \geq \prod_{i=1}^{n} U'_i,$$

  where we denoted $U_i \stackrel{\text{def}}{=} u_i - s_i$ and $U'_i \stackrel{\text{def}}{=} u'_i - s_i$.

Comment. In our proof, we will show that we do not actually need the fairness condition: in this case, it follows from the other three conditions. However, we keep this reasonable condition in the definition, since, as we show later, in the crisis case, it does not automatically follow from the other conditions.

3 Can a Similar Approach Find a Fair Way to Share a Crisis? Unfortunately, No

Discussion. In case of a crisis, when we can no longer maintain the status quo level, fairness means that everyone should contribute, i.e., that we only consider vectors x for which $u_i < s_i$ for all i. In this case, it is reasonable to make similar requirements about preferences as in the case of success – Pareto-optimality, scale-invariance, etc. Unfortunately, in this case, it is not possible to satisfy all these four conditions.

Definition 3. *Let $n > 1$ be an integer. We say that we have a* crisis-related preference relation *if for every vector $s \in \mathbb{R}^n$, we have total pre-order relation $\preceq_s$ on the set of all tuples x_i for which $x_i < s_i$ for all i.*

Proposition 2. *No crisis-related preferences relation is Pareto-optimal, scale-invariant, anonymous, and fair.*

Comment. As we can see from the proof, if we do not impose the fairness condition, then the smaller the product of losses $s_i - u_i$, the better. In this case, there is no best outcome, but we can get as close to the best if we let at least one participant to keep almost everything, i.e., to have $u_i \approx s_i$ – which others will suffer. This is clearly not a fair solution.

4 So What Is a Fair Way to Share a Crisis?

Discussion. The negative result from the previous section shows that we cannot come up with a fair solution if all we know is the current state – the status quo state s. So, to come up with a fair solution, a natural idea is to also take into account the state s' at some previous moment of time.

So, we need a mapping – or maybe several possible mappings – that will, given two vectors s and s', compute the reduced-gain vector u, with $u_i < s_i$. Similarly to the case of sharing a gain, it makes sense to require scale-invariance. Thus, we arrive at the following definition.

Definition 4. *By* crisis-related decision function*, we mean a continuous function $F(s, s')$ that transforms pair of tuples into a new tuple s'' for which $s'' \le s$. We say that the decision function is:*

- scale-invariant *if for every two tuples $c > 0$ and d, $s'' = F(s, s')$ implies $T_{c,d}(s'') = F(T_{c,d}(s), T_{c,d}(s'))$;*
- anonymous *if for every i and j, $s'' = F(s, s')$ implies*

$$\pi_{i,j}(s'') = F(\pi_{i,j}(s), \pi_{i,j}(s')).$$

Proposition 3. *For each crisis-related decision function $F(s, s')$, the following two conditions are equivalent to each other:*

- *the function $F(s, s')$ is scale-invariant and anonymous, and*
- *there exist values $\alpha_+ \ge 0$ and $\alpha_- \ge 0$ for which, for all s and s', the components of the tuple $s'' = F(s, s')$ have the following form: $s''_i = s_i - \alpha_+ \cdot (s_i - s'_i)$ when $s_i \le s'_i$ and $s''_i = s_i - \alpha_- \cdot (s'_i - s_i)$ when $s'_i \le s_i$.*

Discussion.

- When for all i, we have $s'_i \le s_i$, then we only need to use the parameter α_+. The smaller α_+, the better. Thus, in this case, Proposition 3 uniquely determines the optimal strategy: we need to select the smallest possible value α_+.
- Similarly, when for all i, we have $s'_i \le s_i$, then we only need to use the parameter α_-. The smaller α_-, the better. Thus, in this case, Proposition 3 uniquely determines the optimal strategy: we need to select the smallest possible value α_-.

– In the general case, when for some i, we have $s'_i < s_i$ while for other indices j, we have $s_j < s'_j$, we have a whole family of possible solutions: namely, a 1-D family corresponding to Pareto-optimal solutions, i.e., in this case, pairs (α_+, α_-) of values that cannot be both reduced.

5 Proofs

Proof of Proposition 1.

1°. It is easy to see that the Nash's preference relation satisfies the first three conditions. To get the fourth condition, it is sufficient to take the vector $u' = 0.5 \cdot u + 0.5 \cdot \pi_{i,j}(u)$, i.e., a vector in which we replace both values u_i and u_j with their arithmetic average. Indeed, in this case, we keep all the other terms in the product intact and replace the product $U_i \cdot U_j$ with the value $U'_i \cdot U'_j$, where

$$U'_i = U'_j = \frac{U_i + U_j}{2},$$

and one can show that

$$\left(\frac{U_i + U_j}{2}\right)^2 \geq U_i \cdot U_j :$$

indeed, the difference between the left-hand side and the right-hand side is equal to

$$\left(\frac{U_i - U_j}{2}\right)^2 \geq 0.$$

So, to complete the proof of Proposition 1, it is sufficient to prove that every preference relation that satisfies the first three conditions has the Nash's form.

2°. Let us show that because of scale-invariance, we can reduce the family of total pre-orders to a single total pre-order. Indeed, for $c_i = 1$ for all i and $d = -s$, scale-invariance implies that $u \succeq_s u'$ if and only if $U \succeq_0 U'$, where we denoted $U = u - s$ and $U' = u' - s$.

3°. Let us now prove that for every U, we have $\pi_{i,j}(U) \sim_0 U$.

Indeed, since the relation $\sim_0$ is total, we have either $\pi_{i,j}(U) \sim_0 U$, or $\pi_{i,j}(U) \succ_0 U$, or $U \succ_0 \pi_{i,j}(U)$.

– In the second case, anonymity would lead to $\pi_{i,j}(\pi_{i,j}(U)) \succ_0 \pi_{i,j}(U)$, i.e., to $U \succ_0 \pi_{i,j}(U)$, which contradicts to $\pi_{i,j}(U) \succ_0 U$.
– In the third case, anonymity would lead to $\pi_{i,j}(U) \succ_0 \pi_{i,j}(\pi_{i,j}(U))$, i.e., to $\pi_{i,j}(U) \succ_0 U$, which contradicts to $U \succ_0 \pi_{i,j}(U)$.

Thus, the only remaining option is $\pi_{i,j}(U) \sim_0 U$.

4°. Let us now prove that for every vector U and for all i and j, U is equivalent to a vector U' in which both components U_i and U_j are replaced by their geometric mean $\sqrt{U_i \cdot U_j}$.

Indeed, due to Part 3 of this proof, we have

$$(\ldots, \sqrt{U_i}, \ldots, \sqrt{U_j}, \ldots) \sim_0 (\ldots, \sqrt{U_j}, \ldots, \sqrt{U_i}, \ldots).$$

Due to scale-invariance for $d = 0$, $c_i = \sqrt{U_i}$, $c_j = \sqrt{U_j}$, and $c_k = 1$ for all other k, we indeed conclude that

$$(\ldots, U_i, \ldots, U_j, \ldots) \sim_0 (\ldots, \sqrt{U_i \cdot U_j}, \ldots, \sqrt{U_i \cdot U_j}, \ldots).$$

5°. Let us now prove, that every vector U is equivalent to the vector consisting of n geometric means $\overline{U} \stackrel{\text{def}}{=} \sqrt[n]{U_1 \cdot \ldots \cdot U_n}$.

Indeed, we will use Part 4 of this proof to prove, by induction over $i = 0, 1, \ldots, n$, that for each i, the vector U is equivalent to some vector of the type $(\overline{U}, \ldots, \overline{U}, U'_{i+1}, \ldots)$ in which the first i terms are equal to $\overline{U}$.

The base case is easy: for $i = 0$, as the desired vector, we can take the same vector U.

The induction step is as follows. Let us assume that we have such a representation for i:

$$U \sim_0 (\overline{U}, \ldots, \overline{U}, U'_{i+1}, U'_{i+2}, \ldots).$$

Then, due to Part 4 of the proof, we have

$$(\overline{U}, \ldots, \overline{U}, U'_{i+1}, U'_{i+2}, U'_{i+3} \ldots) \sim_0 (\overline{U}, \ldots, \overline{U}, \overline{U}, U''_{i+2}, U'_{i+3} \ldots)$$

as long as $U'_{i+1} \cdot U'_{i+2} = \overline{U} \cdot U''_{i+2}$. So this equivalence holds for

$$U''_{i+1} = \frac{U'_{i+1} \cdot U'_{i+2}}{\overline{U}}.$$

The induction is proven. So, for $i = n$, we get the desired result.

6°. So, due to Part 5, every vector U is equivalent to a vector $(\overline{U}, \ldots, \overline{U})$, where $\overline{U}$ is the n-th root of the product of the values U_i. Thus, every two vectors with the same product are equivalent to each other. Due to Pareto optimality, if the product is larger, the vector is better. So indeed, the preference relation that satisfies the first three condition has the Nash's form.

The proposition is proven.

Proof of Proposition 2. Similarly to the proof of Proposition 1, we can prove that under the first three conditions, every tuple U is equivalent to a tuple $(\overline{U}, \ldots, \overline{U})$, where $\overline{U} = \sqrt[n]{U_1 \cdot \ldots \cdot U_n}$, except for this time $U_i \stackrel{\text{def}}{=} s_i - u_i$. Due to Pareto optimality, the smaller $\overline{U}$, the better. So the only preference relation that satisfies the first three conditions is the relation $u \succeq_s u' \leftrightarrow \overline{U} \geq \overline{U}'$.

To complete the proof, we will show that this preference relation does not satisfy the fourth condition. Indeed, let us take, $s = 0$ and

$$S = \{(-x, -(3 - x), -1, \ldots, -1) : 0 < x < 3\}.$$

Both the status quo state and the set S do not change if we swap participants 1 and 2. For the vector $u = (-2, -1, -1, \ldots, -1)$, we have $\overline{U} = \sqrt[n]{2}$. However, the only outcome which is invariant with respect to the 1-2 swap is $u' = (-1.5, -1.5, -1, \ldots, -1)$. For this outcome, $\overline{U}' = \sqrt[n]{1.5 \cdot 1.5} = \sqrt[n]{2.25} > \overline{U}$. Thus, here $u \succ_s u'$ – and hence, the preference relation violates the fairness condition.

The proposition is proven.

Proof of Proposition 3.

1°. It is easy to show that for each $\alpha_+ \geq 0$ and $\alpha_- \geq 0$, the corresponding function is scale-invariant and anonymous. So, to complete the proof, we need to show that, vice versa, every scale-invariant anonymous function has the desired form.

2°. Due to scale-invariance for $c_i =$ and $d_i = -s_i$, $s'' = F(s, s')$ implies that $s'' - s = F(0, s' - s)$, i.e., that $s'' = F(s, s') = G(s' - s) + s$, where we denoted $G(U) \stackrel{\text{def}}{=} F(0, U)$. Thus, to describe all possible scale-invariant anonymous functions $F(s, s')$, it is sufficient to describe functions $G(U) = F(0, U)$. Since $T_{c,0}(0) = 0$, for this new function, scale-invariance means that for every $c > 0$, if $U' = G(U)$, then $T_c(U') = G(T_c(U))$. In other words, we have $T_c(G(U)) = G(T_c(U))$.

3°. When $U_i = 0$ for some i, then for $c_i = 2$ and $c_j = 1$ for all other j, we have $T_c(U) = U$. Thus, scale-invariance implies that $T_c(G(U)) = G(U)$. For the i-th component of the vector $U' = G(U)$, this means $U'_i = 2U'_i$, thus $U'_i = 0$.

4°. For the values i for which $U_i \neq 0$, we can use $c_i = 1/|U_i|$. Then, the vector $e \stackrel{\text{def}}{=} T_c(U)$ contains only components that are equal to 1, -1, and 0. Let n_- be the number of values equal to -1, n_+ equal to the number of values equal to 1, and n_0 be the number of values equal to 0. For every triple $N = (n_+, n_-, n_0)$ different vectors e corresponding to this case can obtained from each other by permutation. Thus, due to anonymity:

- all n_+ participants i with $e_i = 1$ get the same value t_i which we will denote by $\alpha_+(N)$, and
- all n_+ participants i with $e_i = -1$ get the same value t_i which we will denote by $\alpha_-(N)$, and

By using scale-invariant, we can conclude that the values U'_i have the desired form – with the only exception that now, the values α_+ and α_-, in general, depends on the vector N. To complete the proof, we need to prove that the values α_+ and α_- are the same for al triples N.

5°. Let us pick the value i for which $e_i = 1$. Let us consider a family of vectors that are obtained by multiplying all the values U_j (except for U_i) by some value ε – while U_i remains intact. For all $\varepsilon > 0$, we have the same vector N, so we have the same formulas for U'_i – with the coefficients α_+ corresponding to this vector N. In the limit, s''_i tends to a vector in which there is only one non-zero component, i.e., for which the corresponding vector N' has the form $(1, 0, n-1)$.

Since the function $F(s, s')$ is continuous, the value s''_i corresponding to the limit vector N' should be equal to the limit of the values corresponding to N. For all $\varepsilon > 0$, the value s''_i is the same – corresponding to $\alpha_+(N)$. So, in the limit, this value should remain the same as well. However, for $\varepsilon > 0$, we have $\alpha_+(N)$ corresponding to the original vector N, while in the limit, we have the value $\varepsilon(N')$. Thus, for each vector N, the value $\alpha(N)$ is the same as in the case when only coefficient is different from 0 – so it does not depend on N.

Similarly, the value $\alpha_-(N)$ does not depend on N. The proposition is proven.

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