Why Interval-Valued (and Type-2) Fuzzy Methods Are Often More Effective

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Abstract. Interval-valued and type-2 fuzzy techniques were designed to provide a more adequate representation of expert knowledge than the traditional (type-1) fuzzy techniques. Somewhat unexpectedly, they also often turn out to be more effective even when there is no expert knowledge at all – when we are simply using fuzzy rules to fit experimental data. In precise terms, for the same number of parameters, interval-valued and type-2 systems often provide a better fit for the data and/or better quality control than traditional (type-1) fuzzy techniques. In this paper, we provide a theoretical explanation for this surprising phenomenon.

Keywords: interval-valued fuzzy, type-2 fuzzy, approximation, control

1 Formulation of the problem

In this paper, our objective is to explain why interval-valued and type-2 fuzzy techniques are often more effective than the more traditional (type-1) fuzzy techniques. In order to formulate the phenomenon that we are trying to explain, we need to briefly recall:

- why fuzzy techniques were invented in the first place,
- why interval-valued and type-2 techniques appeared, and
- in what exactly sense these techniques are more effective.

Why fuzzy in the first place: a brief reminder. In the 1960s, Lotfi Zadeh – who at that time was one of the world's main specialists in automated control and one of the most popular textbook on this topic – noticed that in many practical situations, the control that was optimal (based on the existing models of the corresponding systems) was often less effective that control by expert controllers. A natural explanation is that the models on which optimization was based were approximate. These models clearly did not fully adequately reflect the expert knowledge – and the experts were often indeed able to point out some knowledge that the existing models missed.

If this additional knowledge was formulated in precise mathematical terms, it would have been relatively easy to incorporate it into the models used to find the control. The challenge was that this missing knowledge was formulated in terms of imprecise ("fuzzy") words from natural language, like "small" or "about 1". For example, many people know how to drive, and they can describe their driving strategy. For example, if you ask an experience driver what to do when the car is on the freeway going 100 km/h and the car in front at a distance 10 m slows down to 95, most people will correctly reply "brake a little bit". But an automatic controller cannot understand this recommendation, it needs to know with what force – and for how many milliseconds – to press the brakes.

We therefore need techniques for translating such imprecise knowledge into precise computer-understandable terms. Zadeh called such techniques *fuzzy* – and came up with several ideas on how to perform this translation; see, e.g., [1–6].

Traditional (type-1) fuzzy techniques: description and limitations. In the first historically fuzzy techniques provided by Zadeh, each term like "small" was represented by a function $m(x)$ that assigns, to each possible value x of the corresponding quantity, a degree to which this value x satisfies the desired property (e.g., a degree to which x is small). This function is known as a *membership function* ot, alternatively, a *fuzzy set.* The idea is to elicit these degrees from the expert, i.e., to ask the expert to mark this degree, e.g., on the interval $[0, 1]$, so that:

– to values x are which are definitely not small, we assign the degree $m(x) = 0$,
– to values x which are definitely small, we assign the degree $m(x) = 1$, and
– to values x that are somewhat small are assigned degrees $m(x)$ between 0 and 1.

The main limitation of this approach is that:

– similarly to how an expert cannot precisely describe what exactly control value to apply – all he/she can say is that this control can be small.,
– the same expert has similar trouble describing his/her exact degree of confidence that a given control value x is small. F

For example, to many people, 25 C is comfortable while 28 C may be somewhat too hot (especially when there is humidity). But is the degree of hotness 0.8? 0.81? 0.79? This is difficult to distinguish.

To be able to more adequately describe expert knowledge, Zadeh himself suggested alternatives to the type-1 techniques. For example:

– instead of asking the expert to assign a single number $m(x)$, we can ask the expert to assign an *interval* $[\underline{m}(x), \overline{m}(x)]$ of possible values; the resulting techniques are known as *interval-valued* fuzzy techniques;

– alternatively, we can allow the expert to mark his/her degree of confidence that x is small by words from natural language – and then use fuzzy techniques to translate this word into precise terms; in this case, the resulting degree $m(x)$ is not a single number, but a fuzzy subset of the interval $[0,1]$; the resulting techniques are known as *type-2* fuzzy techniques.

These techniques indeed lead to a more adequate representation of expert knowledge; see, e.g., [3].

Fuzzy techniques in situations without expert knowledge. Originally, fuzzy techniques were invented to describe expert knowledge. Later, it turned out that these techniques are also effective in situations when there is no expert knowledge at all. Namely, when we want to fit the data, it is often effective:

– to find the fuzzy rules that best describe this data, and then
– to use these rules to predict how the system will react to different inputs.

This effectiveness makes perfect sense: this is how we learn to do things, we use the data to come up with informal (fuzzy) rules. Annd since we humans are a product of billions of years of improving evolution, this must be an effective strategy.

In situations without expert knowledge, interval and type-2 methods are more effective, but why? Interesting, it turned out that in situations without expert knowledge, interval-valued and type-2 fuzzy techniques often are more effective. Namely, when we compare type-2 and interval or type-2 methods with the same number of parameters, interval and typ2-2 methods provide a more accurate description of the phenomenon and/or a better quality control; see, e.g., [3]. But why?

It is clear why interval-valued and type-2 technique are more effective when the objective is to represent expert knowledge, but why are they more effective in situations when there is no expert knowledge?

What we do in this paper. In this paper, we provide a theoretical explanation for the (somewhat surprising) effectiveness of interval-values and type-2 fuzzy techniques.

2 Our explanation

In order to explain the above puzzling phenomenon, let us briefly recall why, in general, some approximations are more accurate and some are less accurate.

General idea: the more options, the better the approximation. In general, there are many ways to approximate a dependence, there are many possible families of approximating functions. For each family, there is a certain number of different approximating options. Usually, the more options we have, the more accurate the approximation.

We can illustrate this natural idea on the example of approximating numbers from the interval $[0,1]$.

- If we only have one option to approximate – and if we gauge the quality of an approximation by the worst-case absolute value w of the difference between the actual value and its approximation – then the best option is to select $x_1 = 0.5$, then $w = 0.5$.
- If we are allowed 2 options, then the best way is to have $x_1 = 0.25$ and $x_2 = 0.75$, in which case $w = 0.25$.
- In general, if we are allowed n options, then the best idea is to have

$$x_1 = \frac{1}{2n}, x_2 = \frac{3}{2n}, \ldots, x_i = \frac{2i-1}{2n}, \ldots, x_n = \frac{2n-1}{2n},$$

in which case

$$w = \frac{1}{2n}.$$

Consequence: the more parameters, the better the approximation. In practice, each parameter is represented in a computer with some accuracy ε. So, if we denote the width of the range of possible values of this parameter by W, this means that we can have W/ε possible distinguishable values of this parameter.

Thus, if we have p parameters, we have $(W/\varepsilon)^p$ possible options. Hence clearly, the more parameters we have, the more options we have and thus, the more accurate will be the approximation.

How is all this related to type-1 and interval-valued fuzzy techniques. In principle, each degree $m(x)$ can be any number from the interval $[0, 1]$. There are infinitely many real numbers on the interval $[0, 1]$. However, in a computer, we can only represent the degree $m(x)$ with some accuracy. As a result, we have only finitely many possible values of the degree

$$m_1 < m_2 < \ldots < m_{M-1} < m_M,$$

for some M.

Similarly, while there are infinitely many actual values x of the corresponding quantity, in a computer, we can only represent finitely many values

$$x_1 < x_2 < \ldots < x_{N-1} < x_N$$

for some N. Thus, a general membership function can be represented by a finite number of degrees $m(x_1), \ldots, m(x_N)$ each of which takes values from the finite set $\mathcal{M} \stackrel{\text{def}}{=} \{m_1, \ldots, m_M\}$: $m(x_i) \in \mathcal{M}$.

Usual membership functions:

- first increase (in general, non-strictly) from 0 to 1,
- then maybe take the value 1 for some time,
- then decrease from 1 to 0.

So, for some k for which the value $m(x_i)$ is the largest, we have:

$$m(x_1) \le m(x_2) \le \ldots \le m(x_{k-1}) \le m(x_k) \ge m(x_{k+1}) \ge \ldots \ge m(x_N). \quad (1)$$

So, we have N values $m(x_i)$. These values must satisfy the monotonicity constraint (1) for some k.

What happens in the interval-valued case? In this case, for each x, we have an interval $[\underline{m}(x), \overline{m}(x)]$ of possible values of degree $m(x)$, i.e., in effect, two values $\underline{m}(x) \le \overline{m}(x)$. So, when we consider the case of the same number of parameters as with the type-1 representation, we need to only consider $N' = N/2$ possible values of x. Let us denote these values by

$$x'_1 < x'_2 < \ldots < x'_{N'}.$$

So we get the values $\underline{m}(x'_i)$ and $\overline{m}(x'_i)$. The values of each of the lower and upper membership functions $\underline{m}(x)$ are $\overline{m}(x)$ must satisfy conditions similar to (1):

$$\underline{m}(x'_1) \le \underline{m}(x'_2) \le \ldots \le \underline{m}(x'_{k'-1}) \le \underline{m}(x'_{k'}) \ge \underline{m}(x'_{k'+1}) \ge \ldots \ge \underline{m}(x'_{N'}), \quad (2)$$

$$\overline{m}(x'_1) \le \overline{m}(x'_2) \le \ldots \le \overline{m}(x'_{k'-1}) \le \overline{m}(x'_{k'}) \ge \overline{m}(x'_{k'+1}) \ge \ldots \ge \overline{m}(x'_{N'}), \quad (3)$$

as well as the constraint

$$\underline{m}(x'_i) \le \overline{m}(x'_i) \text{ for all } i. \quad (4)$$

Resulting theoretical explanation. We plan to show that for the same N and M, there are more tuples that satisfy the constraints (2)–(4) than tuples that satisfy the constraint (1). As we have mentioned earlier, this will clearly imply that interval-valued fuzzy techniques can lead to a more accurate approximation than type-1 – which is exactly the empirical fact that needs to be explained.

Indeed, for every sequence of values $m(x_i)$ that satisfies the monotonicity constraints (1), we can form the interval-valued approximation in which:

- for $2i \le k$, we have $\underline{m}(x'_i) = m(x_{2i-1})$ and $\overline{m}(x'_i) = m(x_{2i})$, and
- for $2i > k$, we have $\underline{m}(x'_i) = m(x_{2i})$ and $\overline{m}(x'_i) = m(x_{2i-1})$.

One can easily check that in this case, conditions (2)–(4) are automatically satisfied. This means that interval-valued approximation has at least as many approximating options as the type-1 approximation.

However, there are interval-valued approximations that cannot be obtained this way. Indeed, in each scheme obtained this way, when $2i \le k$, we have $\overline{m}(x'_i) \le \underline{m}(x'_{i+1})$. However, this is not always true for interval-valued fuzzy sets: e.g., we can have $\underline{m}(x'_{i+1}) < 1$ but $\overline{m}(x'_i) = 1$. Thus, with the same number of parameters, interval-valued approximation scheme indeed contains more approximating options than the type-1 approximating scheme, and thus, potentially leads to higher approximation accuracy.

Comment. We provided a detailed explanation only for interval-valued techniques, but a similar argument works for general type-2 techniques: in this case, we have even fewer constraints.

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