How to combine subjective intervals: a natural idea

Niklas Winnewisser, Michael Beer, Olga Kosheleva, and Vladik Kreinovich

Abstract In many practical situations ranging from economics to medicine to geosciences, we use expert estimates of different quantities. To get a better understanding of expert's opinion, it is also reasonable to ask the expert for the perceived accuracy of his/her estimate. As a result, each expert estimate looks like 50 ± 10, i.e., it is, in effect, an interval of the type $[40-10, 40+10]$. Often, we ask several experts. In this case, we need to combine several resulting subjective intervals into a single interval that we will use to make a decision. In this paper, we describe a natural way to combine subjective intervals.

1 Formulation of the problem

Need for expert estimates. In many application areas, to make good decisions, it is useful to take into account not only the measurement results, but also expert estimates. Businesses use expert opinion on the future economics situation when making important decisions, medical doctors consult experts if needed, oil companies use expert estimates of the potential oil fields productivity to decide where to invest their efforts, etc.

Need for interval estimates. Some experts provide more accurate estimates, some less accurate ones. When we take the opinions of different expects into account,

Niklas Winnewisser and Michael Beer
Institute for Risk and Reliability, Callinstraße 34, Leibniz University Hannover
30167 Hannover, Germany, e-mail: winnewisser@irz.uni-hannover.de, beer@irz.uni-hannover.de

Olga Kosheleva
Department of Teacher Education, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: olgak@utep.edu

Vladik Kreinovich
Department of Computer Science, University of Texas at El Paso, 500 W. University
El Paso, Texas 79968, USA, e-mail: vladik@utep.edu

it is desirable to know how accurate is their estimates. A natural way to get this information is to ask an expert to provide not only a numerical estimate x, but also a bound Δ on how far, in his/her opinion, the estimate x can be from the actual value of the estimated quantity.

Once we have both x and Δ, we can then conclude that most probably, the actual value of the corresponding quantity is located in the interval $[x-\Delta, x+\Delta]$. We will call such intervals *subjective*.

Need to combine expert estimates. The reason why we ask experts for their estimates is that we need to use these estimates in our decisions. If we have a single numerical estimate x, then we can use it in our decision making.

However, in many important practical situations, to gain more information, we ask several experts to estimate the same quantity. For example, in civil engineering, inspectors regularly inspect roads, bridges, public building, etc., and report on possible cracks and other faults. For each fault, we have several estimates of its location coming from different inspections.

Based on these estimates, we need to come up with a combined estimate that we will use in our decision making. In other words, we need to combine expert estimates – which are either numerical estimates $x_1, \ldots, x_n$ or interval estimates

$$[x_1-\Delta_1, x_1+\Delta_1], \ldots, [x_n-\Delta_n, x_n+\Delta_n],$$

into a single estimate.

How to combine expert estimates? It is desirable to find a natural way to combine several estimates into a single estimate.

What we do in this paper. In this paper, we provide such a natural combination method. To come up with this method, first, in Section 2, we consider the case when we need to combine numerical estimates. An important dynamical version of this problem is considered in Section 3. Then, in Section 4, we consider the case when we need to combine interval estimates which are of the same width. Finally, in Section 5, we consider the general problem of combining subjective intervals.

2 Combining numerical estimates

Problem: reminder. We have n values $x_1, \ldots, x_n$, and we need to combine them into a single estimate x.

Analysis of the problem. There are many different reasons why an expert estimate is different from the actual value of the corresponding quantity. In statistics, it is known that, under some reasonable conditions, if we have different independent factors, then their joint effect is close to Gaussian; the related mathematical result is known as the *Central Limit Theorem*; see, e.g., [2]. We can therefore conclude that the values x_i come from a Gaussian distribution.

Such a distribution is characterized by two parameters – mean and standard distribution. So, to describe the whole sample $x_1, \ldots, x_n$, we need to estimate the mean m and the standard deviation σ. For example, we can use the usual estimates:

$$m \approx \frac{x_1 + \ldots + x_n}{n} \text{ and } \sigma^2 \approx \frac{1}{n-1} \cdot \sum_{i=1}^{n} (x_i - m)^2. \quad (1)$$

Then, we can conclude, with an appropriate confidence, that the actual value x is contained in a k-sigma interval $[x - k \cdot \sigma, x + k \cdot \sigma]$, for some k.

Which value k to select depends on how much confidence we need. Usually, practitioners select $k = 2$, $k = 3$, or $k = 6$: the selection $k = 3$ corresponds to 99.9% confidence, and the selection $k = 6$ corresponds to $1 - 10^{-8}$ confidence.

Resulting method.

- First, we compute the values (1).
- Then, we inform the interested parties that the actual value of the quantity x is most probably contained in the interval $[x - k \cdot \sigma, x + k \cdot \sigma]$, where the selection of the value k depends on what level of confidence we want.

What if some estimates are outliers? Some of the estimates may be completely way off, outliers. To avoid such outliers influencing the decision, a usual idea in statistics is to use robust versions of statistical methods, i.e., versions for which the resulting value does not change much if we simply add an outlier.

For example, we can:

- sort the values x_i into a monotonic sequence $x_{(1)} \le x_{(2)} \le \ldots \le x_{(n)}$,
- dismiss the lowest 5% and the top 5% of the values, and
- perform computations based only on the remaining values.

Comment. A similar modification can be applied to all the other methods described in this paper.

3 Combining numerical estimates: dynamical case

Problem. The above estimate is appropriate when we have a *static* quantity, i.e., a quantity that does not change with time – like the location of a fault. But sometimes, we are also interested in a dynamically changing quantity – e.g., the size of the fault. This size can increase with time, but it can never decrease.

In this case, arithmetic average is not a good option. In the static case, as we have shown in the previous section, it seems natural to use the arithmetic average of all the expert estimates. However, for dynamic quantities, it makes no sense to take the arithmetic average. For example, if we observe sizes 10, 20, and 30 in three consequent moments of time, then it does not make sense to return the average value 20 as the current fault size – since we already observed size 30.

This example may prompt us to suggest to pick up the latest estimate – but this would make sense only if these were exact measurements. Since these are estimates, and estimates are, in general, different from the actual values, the latest estimate may be smaller than the previous one. So what do we do?

Analysis of the problem. Without losing generality, we can assume that the estimates x_i are sorted chronologically: x_1 is the earliest estimate, x_2 is the second earliest, etc. We know that each estimate x_i is approximately equal to the corresponding actual value a_i, and the only thing that we know about these actual values is that they form a non-decreasing sequence:

$$a_1 \le a_2 \le \ldots \le a_n.$$

In this case, a natural idea is to provide better estimates for the values a_i by minimizing the sum of the squares

$$(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2$$

under the above monotonicity condition – and then return the latest estimate a_n as the estimate for the latest value of the quantity (e.g., the latest value of the fault size).

Resulting algorithm. There is a known algorithm for solving this constraint minimization problem; see, e.g., [1].

Since this algorithm is not as widely known as the general Least Squares technique, let us explain this algorithm. On each stage of this iterative algorithm, the integer-valued interval $I \stackrel{\text{def}}{=} \{1,2,\ldots,n\}$ is divided into several non-intersecting subintervals I_1, I_2, $\ldots$, so that the interval I starts with I_1, which is followed by I_2, etc.

We start by diving into n one-point intervals $I_1 = \{1\}$, $I_2 = \{2\}$, etc. For each sub-interval I_j, we compute the arithmetic mean $m(I_j)$ of all the values x_i for $i \in I_j$.

If for some j, we have $m(I_j) > m(I_{j+1})$, then we merge the sub-intervals I_j and I_{j+1} into a single sub-interval, and repeat the procedure again. We stop when no changes are possible, i.e., when $m(I_1) \le m(I_2) \le \ldots$ Then, for each $i \in I_j$ we take $a_i = m(I_j)$.

Numerical example. Let us illustrate this algorithm on a simple example. Let us assume that $x_1 = 10$, $x_2 = 30$, and $x_3 = 20$. We start with the trivial subintervals $I_1 = \{1\}$, $I_2 = \{2\}$, and $I_3 = \{3\}$. Since each of these sub-intervals consists of a single value i, the arithmetic average is simply the corresponding value x_i: $m(I_1) = 10$, $m(I_2) = 30$, and $m(I_3) = 20$.

In this case, $m(I_2) > m(I_3)$, so we merge the sub-intervals I_2 and I_3 into a single sub-interval $I_2 \cup I_3 = \{2,3\}$. Now we have a subdivision into two subintervals: the original subinterval I_1 for which $m(I_1) = 10$ and the new subinterval $I_2' = \{2,3\}$ for which

$$m(I_2') = \frac{30+20}{2} = 25.$$

Now, $m(I_1) = 10 < m(I'_2) = 25$, so the process stops.

In this final division into sub-intervals, index $i = 1$ is contained in the sub-interval I_1, so we take $a_1 = m(I_1) = 10$. Indices $i = 2$ and $i = 3$ are contained in the sub-interval I'_2, so we take $a_2 = a_3 = m(I'_2) = 25$. In particular, as the latest estimate, we take $a_3 = 25$.

4 Combining interval estimates: case when all the intervals have the same width

Problem: reminder. We have n intervals $[x_1 - \Delta, x_1 + \Delta]$, ..., $[x_n - \Delta, x_n + \Delta]$. We need to combine these intervals into a single interval.

Analysis of the problem. In the previous case, we had only one reason why the estimates are different from the actual value – since we human are not perfect. In the current case, we have two reasons for the difference:

- First, as in the previous case, we are not perfect. Even if we had perfect knowledge of the situation, our estimate would still deviate from the actual value. This explains the difference between the estimates x_i and the combined estimate x.
- Second, we do not have perfect information. So, even if we were perfect reasoning and calculating machines, we would not be able to predict the exact value. This explains the $\pm\Delta$ term.

From this viewpoint, all we need to do is combine the numerical estimates x_i – and we already know how to do it – and then add $\pm\Delta$ term to the resulting confidence interval. Thus, we arrive at the following algorithm.

Resulting method.

- First, we compute the values (1).
- Then, we inform the interested parties that the actual value of the quantity x is most probably contained in the interval $[x - k \cdot \sigma - \Delta, x + k \cdot \sigma + \Delta]$, where the selection of the value k depends on what level of confidence we want.

Examples. Let us illustrate the resulting algorithm on simple examples.

The simplest examples is when all intervals are identical, i.e., $x_1 = \ldots = x_n$. In this case, we have $m = x_i$ and $\sigma = 0$, so the combined interval is equal to each of the combines intervals – which makes sense. If all the experts agree that the fault is located in the interval $[x_i - \Delta, x_i + \Delta]$, then this joint interval should be returned as the group estimate.

This was an extreme case, when all intervals are identical. The opposite extreme case is when intervals are disjoint. Let us consider the simplest such situation, when we have two disjoint intervals, i.e., intervals $[x_1 - \Delta, x_1 + \Delta]$ and $[x_2 - \Delta, x_2 + \Delta]$ for which $x_1 + \Delta < x_2 - \Delta$.

If these intervals were guaranteed bounds on the actual value, then from the fact that these intervals do not have a common point we would conclude that our information is inconsistent. However, strictly speaking, there is no inconsistency: experts are *not* claiming that values outside their intervals are impossible, they just say that such values are less probable.

What should we return in this case? We have two similar opinions, we have no reason to prefer one to another, so it makes sense to return an interval that contains both these intervals – and maybe something else. Let us show that this is exactly what we get if we use the proposed method. Indeed, in this case, we have

$$m=\frac{x_1+x_2}{2},\;\; \sigma=\frac{x_2-x_1}{2},$$

so already for $k=1$, we have

$$m-k\cdot\sigma-\Delta=\frac{x_1+x_2}{2}-\frac{x_2-x_1}{2}-\Delta=x_1-\Delta$$

and similarly

$$m+k\cdot\sigma+\Delta=\frac{x_1+x_2}{2}+\frac{x_2-x_1}{2}+\Delta=x_2+\Delta.$$

So indeed, in this case, the combined interval contains both original intervals – in this case, it is actually the smallest interval that contains both given intervals. For $k>1$, the resulting combined interval is even wider, so it contains both original intervals and some additional points – as expected.

This example does not mean that the resulting interval always contains all given intervals. For example, if we have $N>1$ intervals equal to $[x_1-\Delta,x_1+\Delta]$, and only one interval equal to $[x_2-\Delta,x_2+\Delta]$, then for $N\to\infty$, the combined interval tends to $[x_1-\Delta,x_1+\Delta]$. This too seems to be in line with common sense.

5 General case

Problem: reminder. We have n intervals $[x_1-\Delta_1,x_1+\Delta_1]$, $\ldots$, $[x_n-\Delta_n,x_n+\Delta_n]$. We need to combine these intervals into a single interval.

Analysis of the problem. In this general situation, the accuracies Δ_i of different experts are different. So, it makes sense to take this into account when combining the values x_i: namely, we should give more weight to more accurate estimates, for which the value Δ_i is smaller. In statistical terms, instead of using arithmetic average (1) – that corresponds to the case when all combined values have the same standard deviation – let us assume that the standard deviations σ_i are proportional to the values Δ_i, i.e., $\sigma_i=k\cdot\Delta_i$ for some $k>0$.

In this case, the Least Squares method for finding the combined value m means minimizing the following sum:

$$\sum_{i=1}^{n} \frac{(x_i - m)^2}{\sigma_i^2} = \sum_{i=1}^{n} \frac{(x_i - m)^2}{(k \cdot \Delta_i)^2} = \frac{1}{k^2} \cdot \sum_{i=1}^{n} \frac{(x_i - m)^2}{\Delta_i^2}.$$

If we differentiate this expression with respect to the unknown m and equate the derivative to 0, we get the following expression for the estimate m:

$$m = \sum_{i=1}^{n} w_i \cdot x_i, \text{ where } w_i = \frac{(\Delta_i)^{-2}}{(\Delta_1)^{-2} + \ldots + (\Delta_n)^{-2}}. \quad (2)$$

We can now compute the estimates for σ^2 and for Δ by combining, with the same weights, values $(x_i - m)^2$ and Δ_i:

$$\sigma^2 = \sum_{i-1}^{n} w_i \cdot (x_i - m)^2 \text{ and } \Delta = \sum_{i=1}^{n} w_i \cdot \Delta_i. \quad (3)$$

So, we arrive at the following algorithm.

Resulting method.

- First, we compute the values (2)-(3).
- Then, we inform the interested parties that the actual value of the quantity x is most probably contained in the interval $[x - k \cdot \sigma - \Delta, x + k \cdot \sigma + \Delta]$, where the selection of the value k depends on what level of confidence we want.

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