Why midpoint, why radius (half-width): invariance-based numerical characteristics of an interval and how they are related to color vision and color optical computing

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Abstract We rarely have precise knowledge about a physical quantity. Often, the only information that we have about a quantity is an interval. To process this information, we need to be able to represent intervals in a computer. For this purpose, we need to represent an interval by numbers. Usually, the most effective and efficient ways to represent an interval are either to represent it by its endpoints or by its midpoint and radius (half-width). This choice has been partly explained by using natural invariance – with respect to selecting a different measuring unit or a different starting point for measurements. In this paper, we extend this explanation by listing all numerical characteristics of an interval that have such natural invariances, and we list possible applications of our result.

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1 Formulation of the problem

Intervals are ubiquitous. We rarely know the exact value x of a physical quantity. Usually, we only know bounds $\underline{x}$ and $\overline{x}$ for which $\underline{x} \le x \le \overline{x}$ – i.e., in effect, we only know the interval $[\underline{x}, \overline{x}]$ that contains x; see, e.g., [6, 10, 13, 15, 18].

How should we represent an interval in a computer? To make decisions, we need to process information about the physical quantities. Since this information often comes in terms of an interval, we need to process intervals. To process them, we need to represent intervals in a computer. In a computer, we can only store numbers. So, to be able to store intervals, we need to represent an interval by numbers.

A straightforward idea is to store the two endpoints $\underline{x}$ and $\overline{x}$ of the interval. Often, it is useful to represent an interval by its midpoint

$$\widetilde{x} = \frac{\underline{x} + \overline{x}}{2}$$

and its radius – also known as half-width:

$$\Delta = \frac{\overline{x} - \underline{x}}{2}.$$

Natural question. A natural question is: why these characteristics? What other characteristics can we use?

What is known and what we do in this paper. The only related publications that we could find were [2, 3] that explain why using midpoint is effective and efficient in several practical cases. Some of these explanations are based on invariances, one of the main tool in physics (see, e.g., [4, 19]).

In this paper, we analyze all related invariances, and for each of them, describe which characteristics have these invariance properties. We will see that if we impose too few invariance requirements, then we have a class depending on arbitrary functions – i.e., an infinite-dimensional class. However, if we impose enough requirements, we will get a unique characteristic – or a few-parametric family of characteristics.

The structure of this paper is as follows. Definitions and main results are given in Section 2, Section 3 contains proofs of these results. Brief conclusions and possible applications form the last Section 4.

2 Definitions and results

2.1 Definitions

Let us first describe what we mean by a numerical characteristic and what are the reasonable invariance properties that such a characteristic should satisfy.

Definition 1. *By a* numerical characteristic of an interval *(or simply* characteristic*, for short), we mean a mapping m that maps each interval* $[\underline{x},\overline{x}]$ *into a number* $m([\underline{x},\overline{x}])$.

First natural invariance property: shift-invariance. Numerical values of many physical quantities – time, temperature, etc. – depend on the selection of the starting point. When we replace the starting point with a new one which is a units earlier or smaller, then to describe each physical quantity in the new units, we need to add a to the previous value. For example, if we replace Celsius temperature scale (C) with Kevin scale (K), then all the temperatures increase by 273.16: e.g., 20 C becomes 293.16 K. It is reasonable to require that the selection of the representing point should not change if we shift the starting point.

There are two possible interpretations of this requirement:

- the first interpretation is that if we add a to both endpoints, the new representing value should be equal to the result of adding a to the original representing value; this is known as *covariance*;
- the second interpretation is that if we add a to both endpoints, the representing value should remain the same; this is known as *invariance*.

Let us describe these two options in precise terms.

Definition *ShC***.** *We say that a characteristic is* shift-covariant *(ShC, for short) if for every interval* $[\underline{x},\overline{x}]$ *and for every two numbers a and x, once* $x = m([\underline{x},\overline{x}])$*, then* $x + a = m([\underline{x}+a,\overline{x}+a])$.

Definition *ShI***.** *We say that a characteristic is* shift-invariant *(ShI, for short) if for every interval* $[\underline{x},\overline{x}]$ *and for every number a, we have* $m([\underline{x}+a,\overline{x}+a]) = m([\underline{x},\overline{x}])$.

Comment. Following physics, in situations when there is no confusion, we will use the term “invariance” to describe both invariances in the proper sense of this word – and covariances as well.

Second natural invariance property: scale-invariance. Numerical values of a physical quantity also depend on the choice of a measuring unit. If we change the scale, i.e., if instead of the original measuring unit, we use a new unit which is $c > 0$ times smaller, then all the numerical values get multiplied by c. For example, if we replace meters with centimeters, all numerical values are multiplied by 100: e.g., 2 m becomes 200 cm. It is also reasonable to require that the characteristic either does not change under such transformation or change similarly to all the values.

Definition *ScC***.** *We say that a characteristic is* scale-covariant *(ScC, for short) if for every interval* $[\underline{x},\overline{x}]$ *and for every two numbers* $c>0$ *and* x*, once* $x=m([\underline{x},\overline{x}])$*, then* $c\cdot x=m([c\cdot\underline{x},c\cdot\overline{x}])$.

Definition *ScI***.** *We say that a characteristic is* scale-invariant *(ScI, for short) if for every interval* $[\underline{x},\overline{x}]$ *and for every number* $c>0$*, we have* $m([c\cdot\underline{x},c\cdot\overline{x}])=m([\underline{x},\overline{x}])$.

Third natural invariance property: sign-invariance. Numerical values of some quantities also depend on which direction we consider positive and which negative. For example, if x is a coordinate, we can always change its orientation. If x is electric current, we can call positive current negative and vice versa – nothing will change. In this case, each original value x is replaced by the new value $-x$. It is also reasonable to require that the representing point should not change under this transformation, when we consider the interval $[-\overline{x},-\underline{x}]$ instead of the original interval $[\underline{x},\overline{x}]$.

Definition *SiC***.** *We say that a characteristic is* sign-covariant *(SiC, for short) if for every interval* $[\underline{x},\overline{x}]$*, we have* $m([-\overline{x},-\underline{x}])=-m([\underline{x},\overline{x}])$.

Definition *SiI***.** *We say that a characteristic is* sign-invariant *(SiI, for short) if for every interval* $[\underline{x},\overline{x}]$*, we have* $m([-\overline{x},-\underline{x}])=m([\underline{x},\overline{x}])$.

Possible invariance with respect to nonlinear transformations. So far, we have considered transformations described by linear functions. In some cases, however, we have several scales related by a non-linear transformation. For example, the energy of the signal can be described both in unusual energy units or in logarithmic scale – in decibels. What if we consider covariance and invariance with respect to all possible nonlinear transformations – as long as their are preserving the order, i.e., as long as they are continuous and increasing. Under such transformation $F(x)$, an interval $[\underline{x},\overline{x}]$ gets transformed into $[F(\underline{x}),F(\overline{x})]$

Definition *NC***.** *We say that a characteristic is* nonlinear-covariant *(NC, for short) if for every interval* $[\underline{x},\overline{x}]$*, for every continuous increasing function* $F(z)$*, and for every number* x*, once* $x=m([\underline{x},\overline{x}])$*, then* $F(x)=m([F(\underline{x}),F(\overline{x})])$.

Definition *NI***.** *We say that a characteristic is* nonlinear-invariant *(NI, for short) if for every interval* $[\underline{x},\overline{x}]$ *and for every continuous increasing function* $F(z)$*, we have* $m([F(\underline{x}),F(\overline{x})])=m([\underline{x},\overline{x}])$.

2.2 Results for the case when we have a single invariance

Proposition 1.

- *A characteristic is ShC if and only if it has the form* $f(\overline{x}-\underline{x})+\underline{x}$ *for some function* $f(x)$.
- *A characteristic is ShI if and only if it has the form* $f(\overline{x}-\underline{x})$ *for some function* $f(x)$.
- *A characteristic is ScC if and only if it has the form* $f_{\text{sign}(\underline{x})}(\overline{x}/|\underline{x}|)\cdot|\underline{x}|$ *for some functions* $f_{-1}(x)$ *and* $f_1(x)$.

- *A characteristic is ScI if and only if it has the form* $f_{\mathrm{sign}(\underline{x})}(\overline{x}/|\underline{x}|)$ *for some functions* $f_{-1}(x)$ *and* $f_1(x)$.
- *A characteristic is NC if and only if it returns one of endpoints of the interval, i.e.,* $\underline{x}$ *or* $\overline{x}$.
- *A characteristic is NI if and only if it ignores the input and always returns the same constant* C.

Comment. For sign-covariance and sign-invariance, there is no simplifying equivalent form.

2.3 Results for the case when we have two invariances

First, we can show that we do not get any good results if we require both covariance and invariance of the same type: either there are no characteristics that satisfy both properties or the only such characteristic is a constant that does not depend on the interval at all:

Proposition 2.

- *ShC + ShI: no characteristic is both ShC and ShI.*
- *ScC + ScI: the only characteristic that is both ScC and ScI is* $m([\underline{x},\overline{x}]) = 0$ *for all intervals.*
- *SiC + SiI: the only characteristic that is both SiC and SiI is* $m([\underline{x},\overline{x}]) = 0$ *for all intervals.*

Since we cannot meaningfully combine covariance and invariance of the same type, we need to combine two different types:

Proposition 3. *Here are the descriptions of all characteristics that satisfy two invariance properties:*

- *ShC + ScC:* $\alpha \cdot \overline{x} + (1-\alpha) \cdot \underline{x}$ *for some* α.
- *ShC + ScI: no characteristic is both ShC and ScI.*
- *ShC + SiC: midpoint* $0.5 \cdot \underline{x} + 0.5 \cdot \overline{x}$.
- *ShC + SiI: no characteristic is both ShC and SiI.*
- *ShI + ScC:* $k \cdot (\overline{x} - \underline{x})$.
- *ShI + ScI: a constant function.*
- *ShI + SiC: a function that always returns 0.*
- *ShI + SiI:* $f(\overline{x} - \underline{x})$ *for some function* $f(x)$.
- *ScC + SiC:* $f_{\mathrm{sign}(\underline{x})}(\overline{x}/|\underline{x}|) \cdot |\underline{x}|$, *where the functions* $f_\pm(x)$ *satisfy the following two equalities for all z:* $f_1(z) = -f_{-1}(-1/z) \cdot z$ *and* $f_{-1}(1/z) \cdot z = -f_{-1}(z)$.
- *ScC + SiI:* $f_{\mathrm{sign}(\underline{x})}(\overline{x}/|\underline{x}|) \cdot |\underline{x}|$, *where the functions* $f_\pm(x)$ *satisfy the following two equalities for all z:* $f_1(z) = f_{-1}(-1/z) \cdot z$ *and* $f_{-1}(1/z) \cdot z = f_{-1}(z)$.
- *ScI + SiC:* $f_{\mathrm{sign}(\underline{x})}(\overline{x}/|\underline{x}|) \cdot |\underline{x}|$, *where the functions* $f_\pm(x)$ *satisfy the following two equalities for all z:* $f_1(z) = -f_{-1}(-1/z)$ *and* $f_{-1}(1/z) = -f_{-1}(z)$.

- *ScI + SiI:* $f_{\mathrm{sign}(\underline{x})}(\overline{x}/|\underline{x}|)\cdot|\underline{x}|$, *where the functions* $f_{\pm}(x)$ *satisfy the following two equalities for all* z*:* $f_1(z)=f_{-1}(-1/z)$ *and* $f_{-1}(1/z)=f_{-1}(z)$.

Comment. The value $\alpha\cdot\overline{x}+(1-\alpha)\cdot\underline{x}$ is known in decision theory, where it describes decision under interval uncertainty. For decision purposed, this formula was first derived by a Novelist Leo Hurwicz; see, e.g., [5, 9, 11]. For $\alpha=0$ and $\alpha=1$, we get endpoints of the interval; for $\alpha=0.5$, we get the interval's midpoint.

2.4 Results for the case when we have more than two invariances

Since we cannot have both covariance and invariance of the same type, and there are exactly three different types of invariances – shift, scaling, and sign – the only possibility to have more than two invariances is to have three invariances of three different type:

Proposition 4. *Here are the descriptions of all characteristics that satisfy more than two invariance properties:*

- *ShC + ScC + SiC: midpoint* $0.5\cdot\underline{x}+0.5\cdot\overline{x}$.
- *ShC + ScC + SiI: no characteristic has all these three invariances.*
- *ShC + ScI + SiC: no characteristic has all these three invariances.*
- *ShC + ScI + SiI: no characteristic has all these three invariances.*
- *ShI + ScC + SiC: a function that always returns 0.*
- *ShI + ScC + SiI:* $k\cdot(\overline{x}-\underline{x})$.
- *ShI + ScI + SiC: a function that always returns 0.*
- *ShI + ScI + SiI: a constant function.*

3 Proofs

3.1 Proof of Proposition 1

ShC: For $a=-\underline{x}$, we get $m([0,\overline{x}-\underline{x}])=m([\underline{x},\overline{x}])-\underline{x}$. Thus, $m([\underline{x},\overline{x}])=f(\overline{x}-\underline{x})+\underline{x}$, where we denoted $f(x)\stackrel{\text{def}}{=}m([0,x])$.

ShI: For $a=-\underline{x}$, we get $m([0,\overline{x}-\underline{x}])=m([\underline{x},\overline{x}])$. Thus, $m([\underline{x},\overline{x}])=f(\overline{x}-\underline{x})$, where we denoted $f(x)\stackrel{\text{def}}{=}m([0,x])$.

ScC: For $c=1/|\underline{x}|$, we get

$$m\left(\left[\mathrm{sign}(\underline{x}),\mathrm{sign}(\underline{x})\cdot\frac{\overline{x}}{\underline{x}}\right]\right)=\frac{1}{|\underline{x}|}\cdot m([\underline{x},\overline{x}]).$$

If we multiply both sides of this equality by $|\underline{x}|$, we get the desired expression for $f_{-1}(x) = m([-1,x])$ and $f_1(x) = m([1,x])$.

ScI: For $c = 1/|\underline{x}|$, we get

$$m\left(\left[\operatorname{sign}(\underline{x}), \operatorname{sign}(\underline{x}) \cdot \frac{\overline{x}}{\underline{x}}\right]\right) = m([\underline{x},\overline{x}]).$$

Thus, we get the desired expression for $f_-(x) = m([-1,x])$ and $f_+(x) = m([1,x])$.

NC: We can prove this by contradiction. If for some interval $[\underline{x},\overline{x}]$, the value $x \stackrel{\text{def}}{=} m([\underline{x},\overline{x}])$ is different from both endpoints, then we can always design a piece-wise linear monotonic function $F(z)$ for which $F(\underline{x}) = \underline{x}$, $F(\overline{x}) = \overline{x}$, but $F(x) \neq x$. For this function, $[F(\underline{x}),F(\overline{x})] = [\underline{x},\overline{x}]$, so $m([F(\underline{x}),F(\overline{x})]) = m([\underline{x},\overline{x}]) = x$, but $F(x) \neq x$, so the covariance condition is not satisfied. Thus, for any interval $[\underline{x},\overline{x}]$, we have either $m([\underline{x},\overline{x}]) = \underline{x}$ or $m([\underline{x},\overline{x}]) = \overline{x}$. If for some x, we have $m([\underline{x},\overline{x}]) = \underline{x}$, then, since every two intervals can be obtained from each other by a continuous increasing linear transformation $F(z)$, covariance implies that the same equality holds for all the intervals.

NI: Since every two intervals can be obtained from each other by a continuous increasing linear transformation $F(z)$, invariance implies that the value $m([\underline{x},\overline{x}])$ is the same for all the intervals.

The Proposition is proven.

3.2 *Proof of Proposition 2*

ShC + *ShI*: In this case, for all $a \neq 0$, we have both $m([\underline{x}+a,\overline{x}+a]) = m([\underline{x},\overline{x}])+a$ and $m([\underline{x}+a,\overline{x}+a]) = m([\underline{x},\overline{x}])$. Thus, $m([\underline{x},\overline{x}])+a = m([\underline{x},\overline{x}])$, i.e., $a = 0$ – but we assumed that $a \neq 0$. This contradiction shows that this case is not possible.

ScC + *ScI*: In this case, for all $c > 0$ and $c \neq 1$, we have both $m([c\cdot\underline{x},c\cdot\overline{x}]) = c\cdot m([\underline{x},\overline{x}])$ and $m([c\cdot\underline{x},c\cdot\overline{x}]) = m([\underline{x},\overline{x}])$. Thus, $c\cdot m([\underline{x},\overline{x}]) = m([\underline{x},\overline{x}])$. Since $c \neq 1$, this means that $m([\underline{x},\overline{x}]) = 0$.

SiC + *SiI*: In this case, for every interval, we have both $m([-\overline{x},-\underline{x}]) = -m([\underline{x},\overline{x}])$ and $m([-\overline{x},-\underline{x}]) = m([\underline{x},\overline{x}])$. Thus, $-m([\underline{x},\overline{x}]) = m([\underline{x},\overline{x}])$ and thus, $m([\underline{x},\overline{x}]) = 0$.

The proposition is proven.

3.3 *Proof of Proposition 3*

ShC + *ScC*: Since the characteristic is *ShC*, by Proposition 1, it has the form

$$f(\overline{x}-\underline{x})+\underline{x}.$$

For this expression, scale-covariance means that

$$f(c\cdot\overline{x}-c\cdot\underline{x})+c\cdot\underline{x}=c\cdot f(\overline{x},\underline{x})+c\cdot\underline{x}.$$

For $\underline{x}=0$ and $\overline{x}=1$, we get $f(c)=\alpha\cdot c$, where we denoted $\alpha\stackrel{\text{def}}{=}f(1)$. Thus:

$$m([\underline{x},\overline{x}])=f(\overline{x}-\underline{x})+\underline{x}=\alpha\cdot(\overline{x}-\underline{x})+\underline{x}=\alpha\cdot\overline{x}+(1-\alpha)\cdot\underline{x}.$$

ShC + *ScI*: Here, scale-invariance means that

$$f(c\cdot\overline{x}-c\cdot\underline{x}))+c\cdot\underline{x}=f(\overline{x},\underline{x})+\underline{x}.$$

In particular, for $\underline{x}=0$, $\overline{x}=1$, and $c=x$, we get $f(x)=f(1)$, so $f(x)$ is a constant, and the general *ShC* expression turns into $f(1)+\underline{x}$. Substituting this expression into the scale-invariance equality, we conclude that for any $c\neq 1$, we have $f(1)+c\cdot\underline{x}=f(1)+\underline{x}$, i.e., $c\cdot\underline{x}=\underline{x}$. So, for $\underline{x}\neq 0$, we get $c=1$ – but $c\neq 1$. The contradiction shows that this case is not possible.

ShC + *SiC*: Since the characteristic is *ShC*, by Proposition 1, it has the form $f(\overline{x}-\underline{x})+\underline{x}$. For this expression, sign-covariance means that $f(-\underline{x}-(-\overline{x}))+(-\overline{x})=-(f(\overline{x}-\underline{x})+\underline{x})$. Here, $-\underline{x}-(-\overline{x})+(-\overline{x})-=\overline{x}-\underline{x}$, so the above equality has the form $f(\overline{x}-\underline{x})-\overline{x}=-(f(\overline{x}-\underline{x})-\underline{x})$. If we move all the terms containing f to the left side and all the other terms to the right side, we get $2f(\overline{x}-\underline{x})=\overline{x}-\underline{x}$. Thus,

$$f(\overline{x}-\underline{x})=\frac{\overline{x}-\underline{x}}{2},$$

and thus,

$$m([\underline{x},\overline{x}])=f(\overline{x}-\underline{x})+\underline{x}=\frac{\overline{x}-\underline{x}}{2}+\underline{x}=\frac{\underline{x}+\overline{x}}{2}.$$

ShC + *SiI*: Since the characteristic is *ShC*, by Proposition 1, it has the form

$$f(\overline{x}-\underline{x})+\underline{x}.$$

For this expression, sign-invariance means that

$$f(-\underline{x}-(-\overline{x}))+(-\overline{x})=f(\overline{x}-\underline{x})+\underline{x}.$$

Similarly to the previous case, this implies that $f(\overline{x}-\underline{x})-\overline{x}=f(\overline{x}-\underline{x})+\underline{x}$, i.e., that $-\overline{x}=\underline{x}$ for all intervals, but this is clearly not always true. This contradiction shows that a characteristic cannot be both *ShC* and *SiI*.

ShI + *ScC*: Since the characteristic is *ShI*, by Proposition 1, it has the form $f(\overline{x}-\underline{x})$. For this expression, scale-covariance means that $f(c\cdot\overline{x}-c\cdot\underline{x}) = c\cdot f(\overline{x}-\underline{x})$. For $\underline{x}=0$ and $\overline{x}=1$, this implies that $f(c)=k\cdot c$, where we denoted $k \stackrel{\text{def}}{=} f(1)$.

ShI + *ScI*: Since the characteristic is *ShI*, by Proposition 1, it has the form $f(\overline{x}-\underline{x})$. For this expression, scale-invariance means that $f(c\cdot\overline{x}-c\cdot\underline{x}) = f(\overline{x}-\underline{x})$. For $\underline{x}=0$ and $\overline{x}=1$, this implies that $f(c)=f(1)$, i.e., that the function $f(x)$ is simply a constant not depending on the input.

ShI + *SiC*: Since the characteristic is *ShI*, by Proposition 1, it has the form $f(\overline{x}-\underline{x})$. For this expression, sign-covariance means that $f(-\underline{x}-(-\overline{x})) = -f(\overline{x}-\underline{x})$, hence $f(\overline{x}-\underline{x}) = -f(\overline{x}-\underline{x})$ and thus, $f(\overline{x}-\underline{x})=0$.

ShI + *SiI*: Since the characteristic is *ShI*, by Proposition 1, it has the form $f(\overline{x}-\underline{x})$. For this expression, sign-invariance means that $f(-\underline{x}-(-\overline{x})) = f(\overline{x}-\underline{x})$, hence $f(\overline{x}-\underline{x}) = f(\overline{x}-\underline{x})$. This equality clearly holds for any function $f(x)$.

ScC + *SiC*: When $\underline{x}>0$, then $\overline{x}>0$ and thus, the value of the characteristic has the form $f_1(\overline{x}/\underline{x})\cdot\underline{x}$. For the "minus-interval" $[-\overline{x},-\underline{x}]$, the value of the characteristic is $f_{-1}(-\underline{x}/\overline{x})\cdot\overline{x}$. Thus, the *SiC* condition implies that $f_1(\overline{x}/\underline{x})\cdot\underline{x} = -f_{-1}(-\underline{x}/\overline{x})\cdot\overline{x}$. If we divide both sides of this equality by $\underline{x}$ and denote $z \stackrel{\text{def}}{=} \overline{x}/\underline{x}$, then this equality takes a simplified form $f_1(z) = -f_{-1}(-1/z)\cdot z$.

When $\overline{x}<0$, then the "minus-interval" $[-\overline{x},-\underline{x}]$ has positive lower bound, so we get the same equality as when $\underline{x}>0$.

When $\underline{x}<0$ and $\overline{x}>0$, then we get $f_{-1}(|\underline{x}|/\overline{x})\cdot\overline{x} = -f_{-1}(\overline{x}/|\underline{x}|)\cdot|\underline{x}|$. If we divide both sides of this equality by $|\underline{x}|$ and denote $z \stackrel{\text{def}}{=} \overline{x}/|\underline{x}|$, then this equality takes a simplified form $f_{-1}(1/z)\cdot z = -f_{-1}(z)$.

ScC + *SiI*: When $\underline{x}>0$, then $\overline{x}>0$ and thus, the value of the characteristic has the form $f_1(\overline{x}/\underline{x})\cdot\underline{x}$. For the minus-interval $[-\overline{x},-\underline{x}]$, the value of the characteristic is $f_{-1}(-\underline{x}/\overline{x})\cdot\overline{x}$. Thus, the *SiI* condition implies that $f_1(\overline{x}/\underline{x})\cdot\underline{x} = f_{-1}(-\underline{x}/\overline{x})\cdot\overline{x}$. If we divide both sides of this equality by $\underline{x}$ and denote $z \stackrel{\text{def}}{=} \overline{x}/\underline{x}$, then this equality takes a simplified form $f_1(z) = f_{-1}(-1/z)\cdot z$.

When $\overline{x}<0$, then the minus-interval $[-\overline{x},-\underline{x}]$ has positive lower bound, so we get the same equality as when $\underline{x}>0$.

When $\underline{x}<0$ and $\overline{x}>0$, then we get $f_{-1}(|\underline{x}|/\overline{x})\cdot\overline{x} = f_{-1}(\overline{x}/|\underline{x}|)\cdot|\underline{x}|$. If we divide both sides of this equality by $|\underline{x}|$ and denote $z \stackrel{\text{def}}{=} \overline{x}/|\underline{x}|$, then this equality takes a simplified form $f_{-1}(1/z)\cdot z = f_{-1}(z)$.

ScI + *SiC*: When $\underline{x}>0$, then $\overline{x}>0$ and thus, the value of the characteristic has the form $f_1(\overline{x}/\underline{x})$. For the minus-interval $[-\overline{x},-\underline{x}]$, the value of the characteristic is $f_{-1}(-\underline{x}/\overline{x})$. Thus, the *SiC* condition implies that $f_1(\overline{x}/\underline{x}) = -f_{-1}(-\underline{x}/\overline{x})$. If we denote $z \stackrel{\text{def}}{=} \overline{x}/\underline{x}$, then this equality takes a simplified form $f_1(z) = -f_{-1}(-1/z)$.

When $\overline{x}<0$, then the minus-interval $[-\overline{x},-\underline{x}]$ has positive lower bound, so we get the same equality as when $\underline{x}>0$.

When $\underline{x}<0$ and $\overline{x}>0$, then we get $f_{-1}(|\underline{x}|/\overline{x}) = -f_{-1}(\overline{x}/|\underline{x}|)$. If we denote $z \stackrel{\text{def}}{=} \overline{x}/|\underline{x}|$, then this equality takes a simplified form $f_{-1}(1/z) = -f_{-1}(z)$.

ScI + *SiI*: When $\underline{x} > 0$, then $\overline{x} > 0$ and thus, the value of the characteristic has the form $f_1(\overline{x}/\underline{x})$. For the minus-interval $[-\overline{x}, -\underline{x}]$, the value of the characteristic is $f_{-1}(-\underline{x}/\overline{x})$. Thus, the *SiI* condition implies that $f_1(\overline{x}/\underline{x}) = f_{-1}(-\underline{x}/\overline{x})$. If we denote $z \stackrel{\text{def}}{=} \overline{x}/\underline{x}$, then this equality takes a simplified form $f_1(z) = f_{-1}(-1/z)$.

When $\overline{x} < 0$, then the minus-interval $[-\overline{x}, -\underline{x}]$ has positive lower bound, so we get the same equality as when $\underline{x} > 0$.

When $\underline{x} < 0$ and $\overline{x} > 0$, then we get $f_{-1}(|\underline{x}|/\overline{x}) = f_{-1}(\overline{x}/|\underline{x}|)$. If we denote $z \stackrel{\text{def}}{=} \overline{x}/|\underline{x}|$, then this equality takes a simplified form $f_{-1}(1/z) = f_{-1}(z)$.

3.4 Proof of Proposition 4

This proof directly follows from Proposition 3.

4 Conclusions and possible applications

Conclusions. As we can see, the only cases when invariance resulted in uniquely determined characteristics – or at least in a finite-parametric family of characteristics – are:

- either characteristics of the type $\alpha \cdot \overline{x} + (1-\alpha) \cdot \underline{x}$, for which special cases are left endpoint, midpoint, and right endpoint,
- or characteristics of the type $k \cdot (\overline{x} - \underline{x})$ which are proportional to the interval's radius (half-width).

So, invariance indeed justifies the empirical success of using these two classes of characteristics.

Possible applications. The only three characteristics of an interval that are uniquely determined by their invariance properties are the left endpoint, the right endpoint, and the midpoint of the interval. So, if we have an interval and we want to select a few meaningful points on this interval, these three are the points that we should select.

This fact provides an explanation for why in many applications of fuzzy logic (see, e.g., [1, 7, 14, 16, 17, 22]), to describe different possible values of a quantity, we use three terms: small, medium, and large, which correspond exactly to, correspondingly, being close to the left endpoint, being close to the midpoint, and being close to the right endpoint.

This also explains why, from the whole interval of light frequencies corresponding to the visible spectrum, evolution selected three specific wavelengths that we perceive: red, green, and blue, which represent exactly the left endpoint, the midpoint, and the right endpoint of this interval. It has been shown that by using this selection of basic colors and their combinations, we can effectively – and fast – perform analog computations, including computations related to fuzzy techniques; see, e.g., [8, 12, 20, 21].

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