How to Describe Commonsense Implication: Between Conditional Probability and Logical Approach

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Abstract It is well known that logical implication does not always reflect the commonsense understanding of if-then statements. For example, statements like "if $2+2=5$ then witches are flying" are absolutely correct from the viewpoint of formal logic, but make no sense from the commonsense viewpoint. In general, it is difficult to describe commonsense implication in precise terms. This difficulty remains when, instead of considering whether an if-then statement is true or not, we try to estimate the probability that a given if-then statement is true. A recent paper has proposed an empirical formula that captures this commonsense probability reasonably well. In this paper, we provide a theoretical explanation for this empirical formula.

1 Formulation of the problem

General problem. From the commonsense viewpoint, how can we define the probability P of an implication "if A then B"?

A seemingly reasonable idea. From the logical viewpoint, implication

"if A then B"

is equivalent to $B \vee \neg A$. So, at first glance, it may seem reasonable to define this probability as $P(B \vee \neg A)$.

This idea is not in agreement with common sense. It turns out that the above idea is not always in agreement with common sense.

For example, according to this definition:

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- for A = "contact with a measles person" and B = "has measles",
- thus defined probability would be 99.999% since most people do not have such a contact.

However, intuitively, such a high probability would be wrongly interpreted as a practically guaranteed infection – but, in reality, it is not the case.

An alternative approach. Another approach is to interpret the desired probability as the conditional probability

$$P(B\,|\,A)=\frac{P(A\,\&\,B)}{P(A)}. \tag{1}$$

However, this is also not always consistent with common sense.

Empirical evidence. Empirical evidence shows that the commonsense probability is between these two values. The following empirical formula describes the commonsense value of the desired probability (see, e.g., [1]):

$$P(A\stackrel{\alpha}{\to}B)\stackrel{\text{def}}{=}\frac{P(A\,\&\,B)+\alpha\cdot(1-P(A))}{P(A)+\alpha\cdot(1-P(A))}, \tag{2}$$

for some α between 0 and 1.

- For $\alpha=0$, we get the conditional probability.
- For $\alpha=1$, we get the probability of $B\vee\neg A$.
- For α strictly between 0 and 1, we get the value in between conditional probability and logic-motivated probability.

A natural question is: how to explain this empirical formula?

What we do in this paper. In this paper, we provide an explanation for the formula (2).

2 Analysis of the problem and the resulting explanation

Why do we need this probability? When the implication "if A is B" is absolutely true, then A always implies B. The need to estimate the probability comes from the fact that the implication is not always true.

Intuitively, this means that:

- in most (or at least in many) cases, A does imply B, but
- there are cases when A does not imply B.

So it makes sense to focus on these cases, i.e., on the probability

$$P(\neg B\,|\,A)=1-P(B\,|\,A).$$

This conditional probability has the form

$$P(\neg B\,|\,A) = \frac{P(A\,\&\,\neg B)}{P(A)}. \tag{2}$$

Some probabilities are more imprecise, some less: a general idea. In general, it is easier to estimate the probability if we have fewer events. For example, it is usually easier to get statistics about events in El Paso than in the whole state of Texas or in the whole US.

Let us use this general idea to simplify our formula. Since $A\,\&\,\neg B$ implies A, this means that there more cases when A holds than cases when we have $A\,\&\,\neg B$.

Thus, the uncertainty with which we know $P(A\,\&\,\neg B)$ is much smaller than the uncertainty with which we know $P(A)$. In the first approximation, it thus makes sense to ignore the smaller uncertainty and to assume:

- that we know the probability $P(A\,\&\,\neg B)$ exactly,
- while we know the probability $P(A)$ with uncertainty.

This uncertainty means that the actual probability $\overline{P}(A)$ of A may somewhat differ from our estimate $P(A)$.

How can we describe the corresponding uncertainty. Because of the formula of total probability, we have

$$\overline{P}(A) = P(A)\cdot P(E) + P(A\,|\,\neg E)\cdot P(\neg E), \tag{4}$$

where E means that our estimate is correct.

If we denote $\alpha \stackrel{\text{def}}{=} P(\neg E)$, then we get

$$\overline{P}(A) = (1-\alpha)\cdot P(A) + \alpha\cdot P(A\,|\,\neg E). \tag{5}$$

We have no information about $P(A\,|\,\neg E)$, it can be any number from $[0,1]$; so:

- the smallest value of $\overline{P}(A)$ is $(1-\alpha)\cdot P(A)$, and
- the largest is $(1-\alpha)\cdot P(A)+\alpha = P(A)+\alpha\cdot(1-P(A))$.

Thus, the smallest – guaranteed – value of the ratio $P(\neg B\,|\,A)$ is

$$\frac{P(A\,\&\,\neg B)}{P(A)+\alpha\cdot(1-P(A))}. \tag{6}$$

This leads to the desired explanation. As a result, the largest value of the desired conditional probability

$$P(B\,|\,A) = 1 - P(\neg B\,|\,A)$$

is equal to

$$1 - \frac{P(A\,\&\,\neg B)}{P(A) + \alpha \cdot (1 - P(A))} = \frac{P(A) - P(A\,\&\,\neg B) + \alpha \cdot (1 - P(A))}{P(A) + \alpha \cdot (1 - P(A))} =$$

$$\frac{P(A\,\&\,B) + \alpha \cdot (1 - P(A))}{P(A) + \alpha \cdot (1 - P(A))}. \quad (7)$$

This is exactly the empirical formula that we wanted to explain.

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References

1. M. Jahn and M. Scheutz, "Generalizing probabilistic material impication and Bayesian conditionals", *International Journal of Approximate Reasoning*, 2023, Vol. 162, Paper 109021.