On Chaos-Related Approach to Dynamics of Exchange Rates: From-First-Principles Derivation of Embrechts et al. Empirical Model

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Abstract Chaos systems are ubiquitous. In particular, chaotic behavior can be observed in finances. This phenomenon was first discovered in 1993 on the example of an empirical model that described the dynamics of fluctuating exchange rates. In this paper, we provide a from-first-principles derivation of this empirical model.

1 Formulation of the problem

Chaotic systems are ubiquitous. Many real-life phenomena show chaotic behavior – when:

- the corresponding dynamical systems are largely deterministic, but
- due to the systems' complexity, we can only make probabilistic long-term predictions.

Chaotic behavior can also be observed in finance. Finance is one of the areas where chaotic behavior has been observed; see, e.g., [3, 4, 5, 6]. Applications of chaos theory to finance started with a pioneering 1993 book [2] that showed, in particular, that chaotic behavior can be observed in the dynamics of exchange rates S_t when this dynamics is described by an empirical formula

$$S_{t+1} = c_t \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1}, \tag{1}$$

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where c_t are behavioral variables that affect the exchange rates.

Comment. If we take logarithms of both sides of the equality (1), we get linear regression:

$$\ln(S_{t+1}) = \ln(c_t) + \alpha_0 \cdot \ln(S_t) + \alpha_1 \cdot \ln(S_{t-1}).$$

A natural question: how can we explain the chaos-related empirical model (1) for the dynamics of exchange rates?

What we do in this paper. In this paper, we provide the from-first-principles derivation of the formula (1).

2 Analysis of the problem

What we want. We want to predict the exchange rate S_{t+1} at the next moment of time based on the current exchange rate S_t and on the past exchange rates $S_{t-1},\ldots,S_{t-k}$.

Ideal case. In the ideal deterministic case, the future value x_{t+1} of a quantity x is uniquely determined by its current value x_t and its past values $x_{t-1}\ldots,x_{t-k}$. In this case, the only thing we need to do to predict the future values is to find a function $x_{t+1} = f(x_t,x_{t-1},\ldots,x_{t-k})$ that best describes the corresponding dynamics.

Specifics of predicting exchange rate. Of course, for exchange rates, even when we have all the previous values of the exchange rate – and even all possible information about the given country's economy – we cannot uniquely predict the future exchange rate. Indeed, the future exchange rate also depends on the fluctuations of the currency to which we are comparing the analyzed currency – be it US Dollar or European Union's Euro; and these fluctuations cannot be predicted based only on the given country's economy.

When the value of the compared-to currency decreases by a factor of c, then the exchange rate between the analyzed country's currency and the compared-to currency is multiplied by c. Thus, in this case, we cannot have a *single* prediction function $f(x_t,x_{t-1},\ldots,x_{t-k})$, we can only have a *family* of prediction functions

$$\{c \cdot f(x_t,x_{t-1},\ldots,x_{t-k})\}_c.$$

So, the question is: which of such families should we select?

Comment: this does not mean that predictions are impossible. The fact that we cannot predict the future exchange rate does not mean that the formulas – like the formula (1) – are useless. Indeed, the coefficient c is the same for all the countries with similar – and related – economies. Thus:

- while we cannot predict the exchange rate of a *single* country,
- we can predict the *ratio* of the exchange rates of two similar countries.

Let us take into account that we can use different compared-to currencies. We have exchange rates with respect to US dollar, with respect to Euro, with respect to Japanese Yen, etc. It is therefore desirable to have a family of functions that should be useful for predicting *all* these different exchange rates.

Let us describe this requirement in precise terms. Suppose that we have a family of functions that describes the dynamics of the exchange rate with respect to US Dollar. For this family, for each moment of time t, we should have

$$S_{t+1} = c_t \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}). \tag{2}$$

Suppose now that instead of comparing the analyzed currency with US Dollars, we compare it with Euros (or with Japanese Yen). To get an exchange rate between the analyzed currency and Euros, we need to multiply two values: the exchange rate between the analyzed currency and US Dollar, and the exchange C_t rate between US Dollar and the new compared-to-currency:

$$S'_t \stackrel{\text{def}}{=} \frac{\text{analyzed currency}}{\text{Euro}} = \frac{\text{analyzed currency}}{\text{US Dollar}} \cdot \frac{\text{US Dollar}}{\text{Euro}} = S_t \cdot C_t.$$

We consider the case when the relation between the two compared-to currencies is reasonably stable. In this case, all the values C_i are around some constant C, with relative difference from C not exceeding some small number $\varepsilon > 0$:

$$1 - \varepsilon \le \frac{C_i}{C} \le 1 + \varepsilon. \tag{3}$$

In this case, for the new exchange rates S'_t, we should have a similar dependence

$$S'_{t+1} = c'_t \cdot f(S'_t, S'_{t-1}, \ldots, S'_{t-k}), \tag{4}$$

for some value c'_t.

As we have mentioned, the formula (2) holds not only for the given country, but for all the countries with a similar economy. Similarly, the formula (4) should be valid for all these countries – which means that the value c'_t should depend only on the two compared-to currencies, but NOT on the present and past exchange rates specific for each of the analyzed countries.

Now, we are ready to describe this requirement in precise terms.

3 Definitions and the main result

Definition 1. *Let $k \ge 0$ be a natural number. By a k-*prediction function*, we mean a measurable function $f(S_t, S_{t-1}, \ldots S_{t-k})$.*

Definition 2. *We say that a prediction function $f(S_t, S_{t-1}, \ldots S_{t-k})$ is* consistent *if there exists a positive number $\varepsilon > 0$ such that for every tuple*

$$(c_t, C, C_{t+1}, C_t, C_{t-1}, \ldots, C_{t-k})$$

that satisfies the condition (3)*, there exists a value* c'_t *for which:*

- *once we have values* S_i *that satisfy the equality* (2)*,*
- *then the values* $S'_i = C_i \cdot S_i$ *satisfy the equality* (4)*.*

Proposition. *A prediction function is consistent if and only if it has the form*

$$f(S_t, S_{t-1}, \ldots, S_{t-k}) = \alpha \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}. \tag{5}$$

Discussion. In particular, for $k = 1$, we get the desired theoretical explanation for the above-mentioned empirical model. For $k > 1$, we get a natural extension of this model to the case when we take into account that the exchange rate may also dependent on the exchange rate a year ago.

Similarly to the case $k = 1$, if we take logarithms of both sides of the equality (1), we get a linear regression:

$$\ln(S_{t+1}) = \ln(\alpha) + \alpha_0 \cdot \ln(S_t) + \alpha_1 \cdot \ln(S_{t-1}) + \ldots + \alpha_k \cdot \ln(S_{t-k}).$$

Proof.

1°. It is easy to prove that every prediction function of type (5) is consistent.

Indeed, in this case, the equality (2) takes the form

$$S_{t+1} = c_t \cdot \alpha \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}, \tag{6}$$

while the desired equality (4) takes the following form:

$$S'_{t+1} = c'_t \cdot \alpha \cdot (S'_t)^{\alpha_0} \cdot (S'_{t-1})^{\alpha_1} \cdot \ldots (S'_{t-k})^{\alpha_k}. \tag{7}$$

Substituting the expression $S'_i = C_i \cdot S_i$ into the formula (7), we get

$$C_{t+1} \cdot S_{t+1} = c'_t \cdot \alpha \cdot (C_t \cdot S_t)^{\alpha_0} \cdot (C_{t-1} \cdot S_{t-1})^{\alpha_1} \cdot \ldots (C_{t-k} \cdot S_{t-k})^{\alpha_k}. \tag{8}$$

Here, $(C_i \cdot S_i)^{\alpha_i} = C_i^{\alpha_i} \cdot S_i^{\alpha_i}$. Substituting these expressions into the formula (8) and placing together all the terms proportional to the powers of S_i, we conclude that

$$C_{t+1} \cdot S_{t+1} = c'_t \cdot \alpha \cdot C' \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}, \tag{9}$$

where we denoted $C' \stackrel{\text{def}}{=} C_t^{\alpha_0} \cdot C_{t-1}^{\alpha_1} \cdot \ldots \cdot C_{t-k}^{\alpha_k}$. Thus, the desired formula (4) is equivalent to

$$S_{t+1} = \frac{c'_t}{C_{t+1}} \cdot \alpha \cdot C' \cdot S_t^{\alpha_0} \cdot S_{t-1}^{\alpha_1} \cdot \ldots S_{t-k}^{\alpha_k}. \tag{10}$$

The desired equality (10) turns into known equality (6) when

$$\frac{c'_t}{C_{t+1}} \cdot C' = c_t,$$

i.e., when we take

$$c'_t = \frac{C_{t+1} \cdot c_t}{C'}.$$

So, for this c'_t, (2) indeed implied (4).

2°. Let us prove that, vice versa, every consistent prediction function has the form (5).

Indeed, for $C = 1$, substituting $S'_i = C_i \cdot S_i$ into the formula (4), we conclude that

$$C_{t+1} \cdot S_{t+1} = c'_t \cdot f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}). \tag{11}$$

Substituting the expression (2) for S_{t+1} into the formula (11), we conclude that

$$C_{t+1} \cdot c_t \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}) = c'_t \cdot f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}), \tag{12}$$

i.e., equivalently, that

$$f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}) =$$
$$g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}) \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}), \tag{13}$$

where we denoted

$$g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}) \stackrel{\text{def}}{=} \frac{C_{t+1}}{c'_t(c_t, C_t, \ldots, C_{t-k})}.$$

The formula (13) only holds for the values C_i for which $1 - \varepsilon \le C_i \le 1 + \varepsilon$. However, if we have $C''_i = C'_i \cdot C_i$, where $1 - \varepsilon \le C_i, C'_i \le 1 + \varepsilon$, then we get a similar equality:

$$f(C'_t \cdot (C_t \cdot S_t), C'_{t-1} \cdot (C_{t-1} \cdot S_{t-1}), \ldots, C'_{t-k} \cdot (C_{t-k} \cdot S_{t-k})) =$$
$$g(c_t, C'_{t+1}, C'_t, \ldots, C'_{t-k}) \cdot f(C_t \cdot S_t, C_{t-1} \cdot S_{t-1}, \ldots, C_{t-k} \cdot S_{t-k}). \tag{14}$$

Substituting the expression (13) into the right-hand side of the formula (14), we get

$$f(C''_t \cdot S_t, C''_{t-1} \cdot S_{t-1}, \ldots, C''_{t-k} \cdot S_{t-k}) =$$
$$g(c_t, C''_{t+1}, C''_t, \ldots, C''_{t-k}) \cdot f(S_t, S_{t-1}, \ldots, S_{t-k}), \tag{15}$$

where we denoted

$$g(c_t, C''_{t+1}, C''_t, \ldots, C''_{t-k}) \stackrel{\text{def}}{=}$$
$$g(c_t, C'_{t+1}, C'_t, \ldots, C'_{t-k}) \cdot g(c_t, C_{t+1}, C_t, \ldots, C_{t-k}). \tag{16}$$

In other words, we can have a similar formula (13) not only for the tuple of values C_i between $1-\varepsilon$ and $1+\varepsilon$, but also for componentwise products of such tuples.

Similarly, we can the same result (13) for componentwise product of any finite number of such tuples. For each positive number C_i, the value $C_i^{1/n}$ tends to $C_i^0 = 1$ when n tends to infinity. Thus, for any tuple of positive numbers, and for sufficiently large n, we have $1-\varepsilon \le C_i^{1/n} \le 1+\varepsilon$ for all i. For the resulting tuple of n-th order roots, we can get the formula (13) and thus, we can get it for the componentwise product of n such tuples – i.e., to the original tuple of the values C_i.

So, the equality (13) holds for all possible positive tuples S_i and C_i. And it is known (see, e.g., [1]) that all measurable functions that satisfy the equality (13) for all positive tuples have the form (5).

The proposition is proven.

Acknowledgments

This work was supported in part by the National Science Foundation grants 1623190 (A Model of Change for Preparing a New Generation for Professional Practice in Computer Science), HRD-1834620 and HRD-2034030 (CAHSI Includes), EAR-2225395 (Center for Collective Impact in Earthquake Science C-CIES), and by the AT&T Fellowship in Information Technology.

It was also supported by a grant from the Hungarian National Research, Development and Innovation Office (NRDI), by the Institute for Risk and Reliability, Leibniz Universitaet Hannover, Germany, and by the European Union under the project ROBOPROX (No. CZ.02.01.01/00/22 008/0004590).

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