How to Make Fuzzy Estimates Less Subjective: Q-Sort, Intuitionistic Fuzzy, and Color Interpretation of Fuzzy Degrees

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Abstract In many practical applications including pedagogy, it is important to make sure that experts provide similar numerical estimates to similar level of uncertainty – and thus make these numerical values less subjective. To make sure that the expert's fuzzy degrees are less subjective, a natural ideas is to calibrate each expert, i.e., to provide a expert with several common examples of different levels of uncertainty, so that the expert will assign his/her degrees by comparing his/her uncertainty in a given statement with these examples. Due to the seven-plus-minus-two laws, it seems natural to select 7 examples, corresponding to 7 distinguishable degree of certainty. However, interestingly, it turns out that the empirically most effective scheme – known as Q-sort – uses 16 examples instead of 7. In this paper, we show that intuitionistic fuzzy logic provides a natural explanation for the effectiveness of Q-sort.

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1 Formulation of the practical problem

Fuzzy techniques are needed: a brief reminder. In many control and decision-making situations, there are a few best specialists – the best medical doctors who excel in diagnosing and curing diseases, the best pilots, the best machine operators, etc. These specialists are very good, but there are only a few of them: it is not possible, e.g., to make sure that every patient is treated by one of the best doctors. It is therefore desirable to design automated control and decision-making systems that would incorporate the knowledge of the best specialists – and either perform automatically, or provide advice to other specialists.

Most specialists are willing to share their skills and their knowledge, but the problem is that a significant part of this knowledge is described not in precise terms – that are easy to implement in a computer – but by imprecise ("fuzzy") natural-language terms like "small", terms that that are not easy for computer to understand. This problem was recognized by Lofti Zadeh, who proposed a technique for translating such fuzzy knowledge into precise computer-understandable terms. Zadeh called such translation techniques *fuzzy*. In his original fuzzy technique, each fuzzy term is described by assigning, to each possible value x of the corresponding quantity, a degree – e.g., on a scale from 0 to 1 – that describes to what extent x has this property – e.g., to what extent x is small.

Most people do not have trouble describing such degrees: this is how we all evaluate the quality of a service on a scale, this is how students evaluate our teaching. As a result, Zadeh's version of fuzzy technique led practitioners to many successful applications; see, e.g., [2, 6, 9, 11, 12, 20].

Remaining practical problem. One of the remaining problems is that fuzzy degrees are somewhat subjective. Different people describe the same level of satisfaction by different values from the interval $[0,1]$. This is a known phenomenon in polls: e.g., it is known that people from New York City, in general, provide lower degrees than people from the Southern states of the US. Because of this subjectivity, different experts provide different numbers for the same degree of satisfaction. As a result, the decisions made by the fuzzy-based systems depend not only on the quality of the expert's opinion – as we would like to – but also on in what US state the expert lives.

It does not make sense that we may get two different medical decisions based on the similar knowledge of two equally qualified medical doctors from two different geographic areas. It is therefore desirable to make fuzzy degrees less subjective, so that for the fuzzy systems based on similar knowledge of two equally qualified doctors, we would get the same automatic recommendations.

2 Calibration – a natural idea, and how it is actually used for fuzzy degrees

Calibration: a general idea. Measuring instruments are not perfect. For example, a watch may be 2 minutes behind. In this case, to come up with more accurate time values, there is no need to mingle with the mechanism: it is sufficient to add 2 minutes to what the watch shows. Similarly, we can correct the reading if the watch fall behind 3 minutes per day – this is not so common for modern electronic watches, but it was happening in the past. In general, manufacturers of measuring instruments *calibrate* their instruments: they try them on different vales $x_1, \ldots, x_n$ of the corresponding quantity, and if the instrument's reading x'_i is somewhat different from x_i, they add an automatic conversion of each measured value x'_i into x_i.

Let us apply the same idea to fuzzy techniques. It is therefore reasonable to apply a similar procedure for fuzzy. Namely, for each of several values $x_1 < \ldots < x_n$ from the interval $[0,1]$ we can prepare several situations in which most people agree that they correspond to degree x_i, and thus experts may be able to better align relate their numerical degree with the community's choice.

There is no need to select too many such degrees: according to the well-known seven-plus-minus-two law (see, e.g., [10, 13]) people usually classify everything into 5-to-9 categories, on average into 7. Thus, it makes sense to select 7 different degrees on the interval $[0,1]$. It makes sense to have them equally distributed on the interval $[0,1]$, so these degrees are:

$$0/6 = 0,\ 1/6,\ 2/6 = 1/3,\ 3/6 = 1/2,\ 4/6 = 2/3,\ 5/6, \text{ and } 6/6 = 1.$$

This is actually used in applications of fuzzy, the corresponding degrees correspond to Strongly Disagree, Moderately Disagree, Slightly Disagree, Neutral, Slightly Agree, Moderately Agree, and Strongly Agree. To make it easier to analyze – and to explicitly reveal the symmetry between degrees of agreeing and degrees of disagreeing – instead of the interval $[0,1]$, usually, an interval $[-3,3]$ is used. In this scale, the seven degrees are:

$$-3,\ -2,\ -1,\ 0,\ 1,\ 2, \text{ and } 3.$$

Q-sort: a brief description. One area where such an expert calibration is used is pedagogy. In pedagogy, it is definitely important to make sure that when several teachers teach different sections of the same class, their opinion of partially correct answers – and thus, their grades for partially correct answers – are similar. The most widely used example of such calibration is known as *Q-sort*; see, e,g., [3, 4, 5, 14, 18, 19].

Interestingly, instead of 7 examples, it turned out to be more effective to provide an expert with 16 examples:

- one example for each of the degrees 3 and -3;
- two examples for each of the degrees 2 and -2;
- three examples for each of the degrees 1 and -1; and

- four examples for degree 0.

But why? A natural question is: why is this 16-examples techniques most empirically effective? In this paper, we provide a possible answer to this question.

3 Our explanation

Analysis of this problem and the resulting explanation. Intuitively, positive degrees 1, 2, and 3 mean that we have arguments in favor of the corresponding statement, while negative degrees mean that we have arguments in favor of its negation. Crudely speaking, we can say that:

- the degree 0 means that we have no arguments for or against a statement,
- the degree 1 means that we have one argument for,
- the degree 2 means that we have two arguments for – which makes our opinion stronger, and
- the degree 3 means that we have 3 arguments for – the strongest case.

Similarly, the degree $-n$ means that we have n arguments against the statement.

Often, we have both arguments for and against a statement. If we have one argument for and one argument against, they kind of cancel each other, and we remain neutral. In general, if we have n arguments for and m arguments against and $m = n$, they cancel each other and the resulting degree is 0. If we have n arguments for and m argument against, and $n > m$, then m pairs of for-against arguments cancel each other, and the resulting degree corresponds to $n - m$ remaining positive arguments, i.e., to the degree $n - m$. Similarly, if we have n arguments for and m argument against, and $n < m$, then n pairs of for-against arguments cancel each other, and the resulting degree corresponds to $m - n$ remaining negative arguments, i.e., to the degree $-(m - n)$. One can seen that in all three possible cases ($n = m$, $n > m$, and $n < m$), the resulting degree is equal to $n - m$.

From this viewpoint, each of the 7 degrees can come from several possible situations: e.g., 0 can come form no arguments at all or from the case when we have equal number of positive and negative arguments. How many situations correspond to each degree?

- Degrees 3 and -3 correspond to only one situation each:

$$3 = 3 - 0 \text{ and } -3 = 0 - 3.$$

- Degrees 2 and -2 correspond to 2 situations each:

$$2 = 2 - 0 = 3 - 1 \text{ and } -2 = 0 - 2 = 1 - 3.$$

- Degrees 1 and -1 correspond to 3 different situations each:

$$1 = 1 - 0 = 2 - 1 = 3 - 2 \text{ and } -1 = 0 - 1 = 1 - 2 = 2 - 3.$$

- Finally, the degree 0 corresponds to 4 possible situations:

$$0 = 0 - 0 = 1 - 1 = 2 - 2 = 3 - 3.$$

This is exactly what we observe in Q-sort – so this is probably why Q-sort is so efficient: it provides not only a calibration of 7 different degrees, but also a calibration of a more refined distinction between different possible situations leading to the same degree.

This explanation is related to intuitionsitic fuzzy degrees. The above description of different situations corresponding to the same degree is not new: it was exactly the idea that motivated the design, by Krassimir Atanassov, of intuitionistic fuzzy logics; see, e.g., [1, 17]. Atanassov noticed that in the traditional fuzzy technique, the same neutral degree 0.5 corresponds to two different situations: when we know nothing about a statement, and when we have an equal number of arguments for and against this statement.

So, he suggested to use, instead of a single degree, two degrees: a positive degree describing the strength of the arguments for a statement and a negative degree degree describing the strength of the arguments against. From this viewpoint, Q-sort provides a calibration of both positive and negative degrees.

Color interpretation. We can also interpret the above explanation in terms of the color interpretation of fuzzy degrees; see, e.g., [7, 8, 15, 16]. In this interpretation, 0 means white, and 1, 2, and 3 means three basic colors, e.g., sorted by the energy of their waves: $1 =$ red (R) $< 2 =$ green (G) $< 3 =$ blue (B).

Each negative degree $-n$ is naturally interpreted as a complementary color – with which the corresponding color n makes white: $-1 = BG$, $-2 = RB$, $-3 = RG$. The fact that some of these degrees can be obtained by combining different positive and negative degrees corresponds to the fact that the same visual color can be formed in different ways: e.g., white can be obtained by combining two different white colors or by combining one of the 3 basic colors with its complementary color.

This may be a way to extend color interpretation of traditional fuzzy degrees – that was successfully used in many applications (see, e.g., [7, 8, 15, 16] and references therein) – to a similar interpretation of intuitionistic fuzzy degrees.

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References

1. K. Atanassov, *Intuitionistic Fuzzy Sets: Theory and Applications*, Springer-Verlag, Berlin, Heidelberg, 1999.
2. R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.
3. L. Bondurant, J. M. Willburne, D. P. Franz, and J. Young, "Sorting it out", *Mathematics Teacher: Learning & Teaching PK-12*, 2025, Vol. 118, No. 12, pp. 904–910.
4. L. Bondurant and J. R. Young, "Sorting out equity: the Q-sort method in mathematics edication research", *Journal of Urban Mathematics Education*, 2024, Vol. 17, No. 1, Paper 84, doi 10.21423/jume-v17ila651
5. D. P. Franz, J. Willburne, D. Polly, and D. A. Wagstaff, "The teacher action Q-sort: a card-sorting tool for professional learning", *National Council of Supervisors of Mathematics (NCSM) Journal of Mathematics Education Leadership*, 2017, Vol. 18, No. 2, pp. 3–14.
6. G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
7. O. Kosheleva, V. Kreinovich, V. L. Timchenko, and Y. P. Kondratenko, "From Fuzzy to Mobile Fuzzy", *Journal of Mobile Multimedia*, 2024, Vol. 20, No. 3, pp. 651–664.
8. A. Lupo, O. Kosheleva, V. Kreinovich, V. Timchenko, and Y. Kondratenko, "There is still plenty of room at the bottom: Feynman's vision of quantum computing 65 years later", In: Y. P. Kondratenko and A. I. Shevchenko (eds.), *Research Tendencies and Prospect Domains for AI Development and Implementation*, River Publishers, Denmark, 2024, pp. 77–86.
9. J. M. Mendel, *Explainable Uncertain Rule-Based Fuzzy Systems*, Springer, Cham, Switzerland, 2024.
10. G. A. Miller, "The magical number seven plus or minus two: some limits on our capacity for processing information", *Psychological Review*, 1956, Vol. 63, No. 2, pp. 81–97.
11. H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.
12. V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
13. S. K. Reed, *Cognition: Theories and Application*, SAGE Publications, Thousand Oaks, California, 2022.
14. W. Stephenson, *The Study of Behavior: Q-Technique and Its Methology*, University of Chicago Press, Chicago, Illinois, USA, 1953.
15. V. Timchenko, Y. Kondratenko, and V. Kreinovich, "Logical platforms for mobile application in decision support systems based on color information processing", *Journal of Mobile Multimedia*, 2024, Vol. 20, No. 3, pp. 679–698, doi.org/10.13052/jmm1550-4646.2037
16. V. Timchenko, V. Kreinovich, Y. Kondratenko, and V. Horbov, "Effectiveness evaluations of optical color fuzzy computing", In: Y. P. Kondratenko and A. I. Shevchenko (eds.), *Research Tendencies and Prospect Domains for AI Development and Implementation*, River Publishers, Denmark, 2024, pp. 129–151.
17. P. M. Vassilev and K. T. Atanassov, *Modifications and Extensions of Intuitionistic Fuzzy Sets*, "Prof. Marin Drinov" Academic Publishing House, Sofia, Bulgaria, 2019.
18. S. Watts and P. Stenner, *Doing Q Methodological Research: Theory, Method, and Interpretation*, Sage Publications, Thousand Oaks, California, USA, 2012.
19. G. Yoshizawa, M. Iwase, M. Okumoto, K. Tahara, and S. Takahashi, "Q workshop: an application of Q methodology for visualizing, deliberating, and learning contrasting perspectives", *International Journal of Envirnmental and Science Education*, 2016, Vol. 11, No. 13, pp. 6277–6302, https://eric.ed.gov/?id=EJ1115536
20. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.