How to Avoid Unfair Discreteness in Decision Making

Jozo Dujmovic, Vladik Kreinovich, and Olga Kosheleva

Abstract Many decisions – in particular, medical decisions – are based on a discrete (crisp) threshold, such as 100 F threshold for body temperature. As a result, someone whose temperature is 100.0 gets a treatment, but a person with similar symptoms whose body temperature is 99.9 – practically the same – has to wait. This does not seem fair. At first glance, it may sound like such unfairness is inevitable. However, inspired by one of Zadeh's main ideas – that everything is a matter of degree – we started looking for a solution. And indeed, it turned out that by using appropriate recommendations of decision theory, we can avoid such unfair discreteness in decision making.

1 Formulation of the problem

Discreteness in decision making, and why it sounds unfair. In many practical situations, decision are based on a strict threshold. Let us give a simple example. During the Covid pandemic, when the first Covid vaccine appeared, there was not enough doses to vaccinate everybody. So, naturally, the decision was made to provide it to the most vulnerable population – which, for Covid, was elderly people. Thus, the decision was made to provide the available vaccine to those who are at least 75 years old at a certain moment of time t_0.

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At first glance, this may sounds reasonable, but let us consider two folks who were born with a different of one day. On day t_0, the first person is exactly 75 years old, so he gets the vaccine, but the second person is one day short of 75, so he does not get the vaccine. Clearly, the difference between their ages is miniscule, negligible for all practical purposes – but one person gets a much better chance to survive while the second person continues every day to face a then-deadly risk of the Covid infection. This does not sound fair.

This may be an extreme example, but such examples are ubiquitous. In many cases, we have a crisp decision threshold t for making a certain decision. In this case, for very small ε, for two practically identical persons with values $t-\varepsilon$ and $t+\varepsilon$, the decision will be different: the second person gets the benefits, the first person does not. This does not sound fair.

For example, in the Soviet Union – where two of us (Ok and VK) are from – there was a body temperature threshold – similar to the 100 F threshold familiar from Covid – that determined whether a patient get a permission to stay at home for a few days or whether he/she still has to go to work (or to school). So someone with a temperature of 100.0 recovers at home, having some rest, and another person who has the same sneezing, coughing, headache, and all related fun – but a temperature of 99.9 – has to go to school and suffer there. This does not sound fair either.

This is not just about Russia, such examples are ubiquitous in the US as well. There are exact age and/or income thresholds for being eligible for Medicare and different versions of Medicaid, for being eligible for welfare and tax credits; there are often grade point average thresholds for student scholarships, etc. All this not about a specific threshold: the very existence of a threshold is what makes it sound unfair.

Comment. For an extended description and analysis of this problem, see [2, 3, 4].

What many people think, why we do not agree, and what we do in this paper. At first glance, there seems to be not much that we can do: as the saying goes, life is tough. We can change the threshold, but it does not make the situation any less unfair.

This is what many people may think – and what they *actually* think. But our thinking was that there should be a way out. In this thinking, we were motivated by one of the main ideas of Lotfi Zadeh, the idea that led to fuzzy logic (see, e.g., [1, 8, 11, 16, 17, 21]): that everything is a matter of degree. In the illness example, some people are in perfect health, some are very sick, but for most other people, being sick is a matter of degree. For example, in normal life, if you feel a little bit of cold, you take aspirin, go to work or school, and feel better the next day, If you simply have a runny nose and a light headache, but no fever, you do not usually consider yourself sick. But during Covid, if a person had any symptoms like that, he/she was strongly encourage to stay at home and not to risk infecting others.

Based on this idea, after several tries – that we will describe later in this paper – we came up with a natural way to avoid the unfair discreteness in decision making.

2 How can we avoid discreteness in decision making: a general idea

Let us look for similar situations in which discreteness causes unfairness. If two kids have an apple, they can divide into two halves, so that each of them has an equal share. But what if they have a small piece of such candy, so small that you cannot divide it into two halves? In this case, the situation is discrete: either the first kid gets it or the second kid gets it. Both cases sound unfair – but everyone knows that there is a simple solution to make the situation fair: flip a coin. Whoever is selected gets the candy.

This is not only about kids. This is how sometimes adults do at a party, when they run out of food and do not want to order online: someone picked at random should leave the party for a while and get more food (and/or drinks).

In these two cases, all participants get the same probability of being selected – otherwise, the decision will not be fair. But in more complex situations, it make sense to assign different probabilities to different people. For example, the United States would like to encourage more people with high skills to become its permanent residents, but it cannot afford to invited millions who have high skills. The solution is the green card lottery, where everyone with high skills (e.g., a PhD) gets a chance to win, but the probability of being selected depends on the skills level: the higher the level, the higher the probability to be selected.

Resulting idea. So, instead of selecting a threshold – as it is done now in many cases – let us assign a probability to each person. In the above two medical examples, the more vulnerable people or more sick people get higher probability of being selected for a treatment or for the permission to stay at home. If this probability smoothly changes with age or with temperature, this does not sound unfair any more: the probability of a person with temperature 99.9 to stay at home is only slightly smaller than for a person whose temperature is 100.0. There is no discreteness, there is no unfairness.

But how can we implement this idea? How exactly should we select these probabilities? If around 100 degrees, the probability changes too fast, we still get a feeling of unfairness. On the other extreme, if it changes too slow, we may have people with almost normal temperature have a probability to stay at home. If an almost healthy person stays at home and someone who is really sick has, with some probability, to go to work – how is this fair?

To properly implement this idea, we need to describe, in precise terms, what exactly we want, and then try to maximize the corresponding objective function.

Natural restriction on probabilities. Let us denote by V the overall number of vaccines or the overall number of permissions to stay at home that the society can afford. Let us denote by n the number of persons who may qualify for this award, and by p_i, the probability that the i-th person gets this award.

The probability means, crudely speaking, that if we repeat the random assignment a large number N of times, then the person i gets the award approximately

$N \cdot p_i$ times. The overall number of awards allocated during these N iterations is $N \cdot V$. All these awards go to one of the n persons, so we must have

$$N \cdot p_1 + \ldots + N \cdot p_n = N \cdot v. \tag{1}$$

If we divide both sides by a common factor N, we get the following condition:

$$p_1 + \ldots + p_n = V. \tag{2}$$

Of course, all the probabilities must be between 0 and 1:

$$0 \le p_i \le 1, \;\; i = 1, \ldots, n. \tag{3}$$

For example, if we have only one piece of candy (i.e., $V = 1$), then the sum of the probabilities should be equal to 1:

$$p_1 + \ldots + p_n = 1. \tag{4}$$

What we do in the following sections. There are many ways to assign probabilities so that the condition (2) is satisfied. Which assignment should we select?

This is what we analyze in the following sections. We will consider several seemingly reasonable ways to formulate the objective in precise terms, and we will see that, in several cases, we do not get a good solution. Only when we start using a rather sophisticated decision theory recommendations, we are able to come up with a reasonable solution.

3 First try: let us maximize the overall society benefit

Natural idea. We are interested in formulating a society-wide decision. So, it seems natural to use the overall society benefit as the criterion for selecting the probabilities.

Let us describe this idea in precise terms. Let u_i denote the benefit that the i-th person (and thus, the society as a whole) gets when he/she gets the award. So, with probability p_i, the i-th person gets the award u_i and with the remaining probability $1 - p_i$ this person gets 0. The average award for this person is thus equal to

$$p_i \cdot u_i + (1 - p_i) \cdot 0 = p_i \cdot u_i. \tag{5}$$

The overall average award for the society as a whole is the sum of all these individual awards:

$$J \stackrel{\text{def}}{=} p_1 \cdot u_1 + \ldots + p_n \cdot u_n. \tag{6}$$

So, we arrive at the following well-defined optimization problem: among all the tuples $(p_1, \ldots, p_n)$ that satisfy the conditions (2) and (3), we need to select the tuple with the largest possible value of the objective function (6).

Unfortunately, this seemingly natural idea does not help us avoid unfair discreteness. The solution to this optimization problem can be easily found. To describe this solution, without losing generality, let us assume that the persons are sorted in the increasing order of the individual awards:

$$u_1 \leq \ldots \leq u_n. \tag{7}$$

In this case, if $p_n < 1$, then we can decrease the probability of some other person and correspondingly increase p_n – as a result, the overall society award will increase. So, when the society award is maximized, we cannot have $p_n < 1$ – so we must have $p_n = 1$. Similarly, we conclude that $p_{n-1} = 1$, etc. – until we reach the number V of awards. So, in this solution, V top awardees get everything with the probability 1, while everyone else gets 0. In other words, we here have a clear threshold – as before.

Comment. This conclusion should not be surprising to those who know different optimization techniques. Indeed, the above problem is a special case of so-called *linear programming* problem (see, e.g., [20]), when we want to maximize (or minimize) a linear function under constraints which are linear equalities or inequalities.

It is known that in such problems, the maximum is attained at one of the vertices of the corresponding set of possible tuples, i.e., at a point where we have exactly n equalities – either the original equalities or equalities corresponding to the original inequalities.

In our case, we have one original equality that needs to be satisfies: the equality (2). Thus, we have to satisfy $n-1$ equalities corresponding to inequalities (3). Each inequality $0 \leq p_i \leq 1$ leads to two possible equalities: $p_i = 0$ and $p_i = 1$. Of course, these two equalities cannot be both true: only one of them can be true. So, the remaining $n-1$ inequalities correspond to $n-1$ persons i for all of whom we will have $p_i = 0$ or $p_i = 1$. The sum of all these probabilities is an integer, and since V is an integer, the remaining probability

$$p_{i_0} = V - p_1 - \ldots - p_{i_0-1} - p_{i_0+1} - \ldots - p_n \tag{8}$$

should also be an integer – and thus, since this is a probability, either 0 or 1.

But why? Overall society benefit seems to be a natural idea. This is, e.g., how we evaluate the quality of a country's economy – by measuring its Gross Domestic Product (GDP). So what is wrong with that?

What is wrong can be described by an old joke: What happens when Bill Gates walks into a bar? On average, everyone in the bar becomes a millionaire. This is a joke, but if, for example, Bill Gates or any other billionaire moves into El Paso, a city with 700 thousand people, then on average, everyone becomes at least $1,400 richer – but that will not affect the level of living of all the other citizens. So, what can we do?

4 Second try: let us use decision theory recommendations

What decision theory recommends. It is well known that the overall society benefit – or, equivalently, the average income – is not a goof criterion of the society's prosperity. This problem has been thoroughly analyzed by decision theory (see, e.g., [6, 7, 9, 10, 14, 15, 18]), where it was shown that under some reasonable conditions, the only way to make group decisions is to maximize the product $e_1 \cdot \ldots \cdot e_n$ of the expected utilities $e_1, \ldots, e_n$. This criterion was first formulated by the Nobelist John Nash, it is known as *Nash's bargaining solution* [10, 12, 13].

Comment. It is worth mentioning that Nash's solution is in good accordance with fuzzy techniques. Indeed, we want to make sure that the first person is happy *and* that the second person is happy, etc. For each person, it is reasonable to describe his/her degree of happiness by its expected utility.

How can we deal with "and"? In fuzzy logic, there are many ways to represent "and". The two most widely used ways – introduced in the very first Zadeh's paper [21] on fuzzy logic – is either to use minimum or to use product. The problem with using minimum for group decision making is that different people may have different scales for describing happiness – it is difficult to compare utilities of dofferent people. So, a natural conclusion is to use the product – which is exactly what Nash's bargaining solution is about.

Let us apply the decision theory recommendations. The i-th person gets utility u_i with probability p_i and 0 with the remaining probability $1 - p_i$. So, as we have mentioned earlier, this person's expected utility is equal to $e_i = p_i \cdot u_i$. The maximized product J of these expected utilities is equal to

$$J = e_1 \cdot \ldots \cdot e_n = p_1 \cdot u_1 \cdot p_2 \cdot u_2 \cdot \ldots \cdot p_n \cdot u_n. \tag{9}$$

We want to find the probabilities p_i for which this product attains the largest possible value.

Here, the utilities u_i do not depend on the probabilities – with respect to these probabilities, they are constants. It is well known that if we divide all the values of the objective function by the same constant

$$J(p_1, \ldots) \to J'(p_1, \ldots) \stackrel{\text{def}}{=} c \cdot J(p_1, \ldots), \tag{10}$$

larger values remain larger and smaller values remain smaller, and the maximum is attained for the same inputs. In particular, for $c = 1/(u_1 \cdot \ldots \cdot u_n)$, this means that to find the values p_i that maximize the product J of expected utilities, it is equivalent to find the values p_i that maximize the function

$$J' = p_1 \cdot \ldots \cdot p_n \tag{11}$$

under the constraints (2) and (3). By applying the Largange multiplier method, we can reduce this constraint optimization problem to the unconstraint optimization of the expression

$$p_1 \cdot \ldots \cdot p_n + \lambda \cdot (p_1 + \ldots + p_n - 1), \tag{12}$$

for some parameter λ known as the Lagrange multiplier. To find the values p_i that maximize this expression, we can differentiate it with respect to each unknown p_i and equate the derivative to 0. Then, we get

$$p_1 \cdot p_2 \cdot \ldots p_{i-1} \cdot p_{i+1} \cdot \ldots p_n + \lambda = 0, \tag{13}$$

i.e., equivalently,

$$\frac{C}{p_i} + \lambda = 0, \tag{14}$$

where we denoted $C \stackrel{\text{def}}{=} p_1 \cdot \ldots \cdot \ldots p_n$. So, we can conclude thatr $p_i = -C/\lambda$. Neither C not λ depend on i, so we conclude that in the optimal solution, we should have $p_1 = \ldots = p_n$.

This is even less fair than using the threshold. So, our conclusion is that everyone should get the same award. Healthy people should get the same change of staying at home as very sick one, the most vulnerable person should get the same chance of getting the vaccine as the least vulnerable one. This does not sound fair at all.

But why? We want to come up with a solution that is the best for the society. But in our analysis, we implicitly assumed that every person only cares about him/herself. This is not realistic. If a person has enough food for him/herself but the whole society collapses, this person will not be able to survive for long. We all depend on the society, we all benefit if the society prospers and we all suffer if the society collapses. So, to make our recommendation adequate, we need to take into account that people do care about society as a whole.

5 A more realistic application of the decision theory recommendations does help to avoid unfairness

How can we take society into account? As we have mentioned, a person's utility U_i – that describe his/her preferences – depend not only on the expected gain $e_i = p_i \cdot u_i$ of this person, but also on the overall society gain

$$g = p_1 \cdot u_1 + \ldots + p_n \cdot u_n. \tag{15}$$

This dependence is usually relatively small: some people are willing to sacrifice everything for the sake of others, but such heroes are rare. Since the dependence is small, it makes sense to use the usual technique actively used in physics [5, 19] and other application areas: expand the dependence of U_i on g and keep only linear terms in this expansion. As a result, we get the following expression:

$$U_i = p_i \cdot u_i + b \cdot g = p_i \cdot u_i + b \cdot (p_1 \cdot u_1 + \ldots + p_n \cdot u_n), \tag{16}$$

for some coefficient b. Then, Nash's bargaining solution means that we maximize the product $J = U_1 \cdot \ldots \cdot U_n$ of all these utilities.

Let us find the optimal arrangement of probabilities. Maximizing the function J is equivalent to maximizing its logarithm

$$\ln(J) = \ln(U_1) + \ldots + \ln(U_n). \tag{17}$$

According to the Lagrange multiplier method, maximizing this expression under the constraint (2) is equivalent to an unconstraint optimization of the following objective function:

$$\ln(U_1) + \ldots + \ln(U_n) + \lambda \cdot (p_1 + \ldots + p_n - 1) =$$
$$\sum_{i=1}^{n} \ln\left(p_i \cdot u_i + b \cdot \sum_{j=1}^{n} p_j \cdot u_j\right) + \lambda \cdot \left(\sum_{i=1}^{n} p_i - 1\right). \tag{18}$$

Differentiating this expression with respect to p_i and equating the result to 0, we get the following equality:

$$\frac{u_i + b \cdot u_i}{p_i \cdot u_i + b \cdot g} + \lambda = 0. \tag{19}$$

If we multiply both sides by the denominator, we get the following equality:

$$u_i(1+b) + \lambda \cdot p_i \cdot u_i + \lambda \cdot b \cdot g = 0. \tag{20}$$

This is a linear equation with respect to the unknown p_i. If we move all the terms not depending on p_i to the other side, we get:

$$(\lambda \cdot u_i) \cdot p_i = -u_i \cdot (1+b) - \lambda \cdot b \cdot g. \tag{21}$$

Thus,

$$p_i = A - \frac{B}{u_i}, \tag{22}$$

where we denoted

$$A \stackrel{\text{def}}{=} -\frac{1+b}{\lambda} \text{ and } B \stackrel{\text{def}}{=} b \cdot g. \tag{23}$$

Of course, due to the constraints (3), this formula has to be limited to the interval $[0,1]$, so we arrive at the following conclusion.

Resulting formula. When the value of the award for each of n persons is u_i, and we have V awards, then, in the optimal distribution, the probability p_i of the person i getting the award is equal to

$$p_i = \min\left(1, \max\left(0, A - \frac{B}{u_i}\right)\right), \tag{24}$$

i.e.,

- to $p_i = 0$ when $u_i \le B/A$,

- to $p_i = 1$ when $u_i \geq B/(A-1)$, and
- to $p_i = A - B/u_i$ when $B/A < u_i < B/(A-1)$.

The values A and B must be determined by the condition (2) and the condition

$$B = 1 + b \cdot \sum_{i=1} p_i \cdot u_i, \tag{25}$$

where b is the parameter that describes to what extend an average person's utility depends on the overall society gain.

Analysis of the formula. When the potential gain u_i is low, we get $p_i = 0$, i.e., we do not allocate this award to the person. When the potential gain is large we allocate the award with probability 1. In the intermediate cases, we allocate the award with some probability that continuously changes with the award. This way, we avoid discreteness and thus, we avoid the resulting unfairness: the probability that a person with temperature 99.9 will be allowed to stay at home is only slightly smaller than the probability of a person whose temperature is 100.0.

Future work: how this formula can be made even more realistic. In our analysis, we only took into account the benefit of the award for each individual person. To make a more accurate decision, we need to also take into account the relation between people. For example, if a person had small kids – or elderly parents – whose welfare depends on this person, then the gain of this person getting the vaccine will be higher than for a lonely person of the same age.

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References

1. R. Belohlavek, J. W. Dauben, and G. J. Klir, *Fuzzy Logic and Mathematics: A Historical Perspective*, Oxford University Press, New York, 2017.

2. J. Dujmovic, *Soft Computing Evaluation Logic: The LSP Decision Method and Its Applications*, Wiley and IEEE Press, Hoboken, New Jersey, 2018.
3. J. Dujmovic, *Graded Logic: Modeling Human Commonsense Reasoning in Decision-Making and AI*, Springer, Cham, Switzerland, 2025.
4. J. Dujmovic, "Concepts and Applications of Graded Logic", Plenary Talk at the 2026 Annual Conference of the North American Fuzzy Information Processing Society NAFIPS 2026, El Paso, Texas, March 14–16, 2026.
5. R. Feynman, R. Leighton, and M. Sands, *The Feynman Lectures on Physics*, Addison Wesley, Boston, Massachusetts, 2005.
6. P. C. Fishburn, *Utility Theory for Decision Making*, John Wiley & Sons Inc., New York, 1969.
7. P. C. Fishburn, *Nonlinear Preference and Utility Theory*, The John Hopkins Press, Baltimore, Maryland, 1988.
8. G. Klir and B. Yuan, *Fuzzy Sets and Fuzzy Logic*, Prentice Hall, Upper Saddle River, New Jersey, 1995.
9. V. Kreinovich, "Decision making under interval uncertainty (and beyond)", In: P. Guo and W. Pedrycz (eds.), *Human-Centric Decision-Making Models for Social Sciences*, Springer Verlag, 2014, pp. 163–193.
10. R. D. Luce and R. Raiffa, *Games and Decisions: Introduction and Critical Survey*, Dover, New York, 1989.
11. J. M. Mendel, *Explainable Uncertain Rule-Based Fuzzy Systems*, Springer, Cham, Switzerland, 2024.
12. J. Nash, "The bargaining problem", *Econometrica*, 1950, Vol. 18, No. 2, pp. 155–162.
13. H. P. Nguyen, L. Bokati, and V. Kreinovich, "New (simplified) derivation of Nash's bargaining solution", *Journal of Advanced Computational Intelligence and Intelligent Informatics (JACIII)*, 2020, Vol. 24, No. 5, pp. 589–592.
14. H. T. Nguyen, O. Kosheleva, and V. Kreinovich, "Decision making beyond Arrow's 'impossibility theorem', with the analysis of effects of collusion and mutual attraction", *International Journal of Intelligent Systems*, 2009, Vol. 24, No. 1, pp. 27–47.
15. H. T. Nguyen, V. Kreinovich, B. Wu, and G. Xiang, *Computing Statistics under Interval and Fuzzy Uncertainty*, Springer Verlag, Berlin, Heidelberg, 2012.
16. H. T. Nguyen, C. L. Walker, and E. A. Walker, *A First Course in Fuzzy Logic*, Chapman and Hall/CRC, Boca Raton, Florida, 2019.
17. V. Novák, I. Perfilieva, and J. Močkoř, *Mathematical Principles of Fuzzy Logic*, Kluwer, Boston, Dordrecht, 1999.
18. H. Raiffa, *Decision Analysis*, McGraw-Hill, Columbus, Ohio, 1997.
19. K. S. Thorne and R. D. Blandford, *Modern Classical Physics: Optics, Fluids, Plasmas, Elasticity, Relativity, and Statistical Physics*, Princeton University Press, Princeton, New Jersey, 2021.
20. R. J. Vanderbei, *Linear Programming: Foundations and Extensions*, Springer, New York, 2014.
21. L. A. Zadeh, "Fuzzy sets", *Information and Control*, 1965, Vol. 8, pp. 338–353.