(Generalized) Z-numbers as foundations of type-2 (and other) generalizations of fuzzy sets and as reflections of physical reality – and how all this can potentially speed up computations

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Abstract We show that Z-numbers – and their natural generalizations – form a natural foundation for type-2 and other generalizations of traditional fuzzy techniques. We also show a natural analogy between (generalized) Z-numbers and physics. This analogy provide a possible path for possible future developments in physics, developments that can potentially lead, in particular, to faster computations.

1 Z-numbers (and their natural generalizations): a brief reminder

Need for computers. Many practical situations are too complex for humans to make good decisions. In this case, we use computers to process data and help us make decisions.

Computers process data by transforming everything into numbers and processing these numbers.

Need for expert knowledge. In many such situations, be it control, medicine, finance, etc., data from measurements is not enough to make a good quality decision, we also need to use human expertise.

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Expert knowledge is often imprecise ("fuzzy"). Experts often use imprecise ("fuzzy") natural language to describe their knowledge.

A natural idea is to ask the expert to provide a numerical estimate of the corresponding quantity x. Sometimes, the expert provides a single estimate, i.e., a single number x. This happens, e.g., when this expert is absolutely confident about this value. However, such situations are rare. In many other cases, the expert can provide several different possible values.

Need for fuzzy techniques. The reason why the expert provides several possible values is that the expert is not 100% confident in his/her estimate. To better take the related degree of confidence into account, it is desirable to ask the expert to gauge his/her level of certainty in each estimate.

To use these estimates in a computer, it is desirable to have this estimate formulated in terms of the native language of computers – namely, in terms of numbers. So, we have some numbers x with numerical degrees of confidence y attached to them. These pairs x-y form what is known as a *fuzzy set*; see, e.g., [3, 11, 16, 19, 21, 24].

Need for Z-numbers. However, this is not all. Just like the expert is not confident is his/her numerical estimate x, the same expert is not 100% confident in his/her numerical estimate y of his/her degree of certainty. So, a natural idea is to ask the expert to provide a numerical value z for his/her degree of confidence y. In this case, we have a triple x-y-z. The part y-z is what is called a *Z-number*; see, e.g., [1, 2, 14, 25].

The value z is also not coming with an absolute confidence, so we can also ask the expert about the degree of confidence t in z, so we have a quadruple x-y-z-t, where the part y-z-t is a natural *generalization of Z-numbers*. To get an even more accurate description of the expert knowledge, we can add more terms to such chains x-y-z-t-$\ldots$-u, with y-z-t-$\ldots$-u a more general generalization of Z-numbers.

2 Many extensions of traditional fuzzy sets can be naturally described in terms of (generalized) Z-numbers

Let us show that most known extensions of the original fuzzy set idea can be described in terms of such chains – i.e., in terms of generalized Z-numbers.

Case of exact knowledge. In some cases, all we have is a single numerical expert estimate x.

Case of set-va;ued uncertainty. When the expert provides several possible values – that form a set $\{x_1,\ldots,x_n\}$ – we get *set-valued uncertainty* and important particular case of *interval uncertainty*; see, e.g., [10, 13, 15, 18].

Case of fuzzy uncertainty. To get a better understanding of the expert knowledge, an expert can provide his/her degree of confidence y in each possible value x. In this case, we have a set of chains $\{x_1\text{-}y_1, \ldots, x_n\text{-}y_n\}$, in which case:

- all x_i's are different, and
- for each x_i, there is only one chain x_i-y with this value x_i.

This corresponds to the traditional *fuzzy uncertainty*.

Comment. It is important to mention that this formulation is somewhat different from the usual definition of a fuzzy set, where we assign some degree to all possible values of x. From this viewpoint, our formulation is closer to practice: in reality, we can only ask finitely many questions to the expert, so this finite list is all we get after the elicitation.

In applications of the traditional fuzzy approach, the values y corresponding to other x are usually obtained by interpolation – i.e., by using linear interpolation to get triangular and trapezoidal membership functions.

Case of set-valued fuzzy uncertainty. To describe the expert's knowledge even better, we can allow the expert to provide several possible estimates y for the same x. Such set of pairs – in which we may have $x_i = x_j$ for some $i \neq j$ – correspond to *set-valued fuzzy* techniques – of which interval-values techniques are most known and most used; see, e.g., [16].

Case of type-2 fuzzy techniques. To get an even better description of the expert's knowledge, we can ask the expert to provide his/her degree of certainty z_i for each pair x_i-y_i. In this case, the expert knowledge is described as set of triples x_i-y_i-z_i – with the condition that for every pair x-y, there is only one corresponding triple. This corresponds to *type-2 fuzzy* techniques.

Case of type-3 fuzzy techniques. We can go further and allow several possible z's for each pair x-y. The resulting set of triples corresponds to *set-valued type-3 fuzzy* techniques; see, e.g., [4, 5, 6, 7, 12]. And if we add degree t to all these triples, we get a set of quadruples x_i-y_i-z_i-t_i with the condition that for each triple x-y-z, there is only one t. This corresponds to the general type-3 fuzzy techniques, etc.

Interval versions of all these uncertainties. In addition to numbers, an expert can provide intervals – so we get an interval-values generalizations of all these examples.

3 How is this related to physical reality

There is a natural relation of all this to physics, namely, with the progression from classical (non-quantum) physics to different versions of quantum physics.

Classical (non- quantum) physics: a natural analogue of the precise knowledge. In the classical (non-quantum) physics, it we know the current state of the word, we can uniquely determine its future state, i.e., we can uniquely determine the value x of each physical quantity that characterizes this future state; see, e.g., [8, 23] for more details on all physical references.

The problem of free will. Classical physics describes many physical phenomena very accurately, but its deterministic character contradicts to the obvious fact that we human have a free will – we can make choices that affect the future state of the world.

Comment. It should be mentioned that, according to Einstein's biography [9], the great physicist seriously believed that free will is an illusion, and that our actions are pre-determined by the initial state of the Universe. He did not just say it: he acted upon this belief. For example, he loved to go yachting on his own – although he could not swim. When his friends got worried about it, he explained that whether a person drowns or not is pre-determined by the initial state of the universe, so no action can change this.

Well, great people have the bright to be somewhat eccentric, but this is clearly not what commons sense tells us.

But is there a free will? Few researchers – Einstein famously included – have believed that free will is an illusion and all our decisions are pre-determined by the initial state of the universe, but most of us are convinced by the clear empirical evidence of free will.

Free will is a natural analogue of set-valued uncertainty. Such non-determinism is not yet reflected in physical theories – that is what we talk about physical reality, not physical theories – but from the physical viewpoint, it makes sense to consider different possible future values, i.e., a set $\{x_1,\ldots,x_n\}$ instead of a single value x. In the analogy with expert knowledge, this corresponds to set-theoretic uncertainty.

Without an explicit taking freedom of will into account, there will be a gap in our natural scheme and natural analogy. So, our analogy can serve as an additional argument in favor of the free will – just like gaps in Mendeleev periodic table turned out to be existing new elements (much rarer that others but still existing).

Quantum physics as a natural analogue of fuzzy uncertainty. In quantum physics, in general, different values x_i can appear, but with different probabilities y_i – which means that when we repeat the same experiment many times, each outcome x_i appears with frequency close to y_i. In this description, the prediction can be described as a set of pairs x_i-y_i for which there is only one pair for each x_i – which corresponds to fuzzy uncertainty.

Of course, in general, degree of certainty is not the same as probability, but probability is one of the possible natural interpretations of numerical degree of certainty – the proportion of statements with the same degree of certainty that eventually turned out to be true.

Quantum physics does not solve the problem of free will. From the viewpoint of free will, free will does not mean that we have a fixed proportion of decisions. So, in different situations, we may have different probabilities. So, we get a general set of pairs x_i-y_i, where we can have $x_i = x_j$ for some $i \neq j$. This corresponds to set-valued uncertainty.

Secondary quantization is a natural analogue of set-values fuzzy techniques. The above describes so-called *primary quantization*, where fields that describe interaction – such as electromagnetic field – are fixed, and only particles show probabilistic behavior. In reality, fields are also quantum objects, and their variation changes the probabilities of different observations. So, here we have triples x_i-y_i-z_i, where z_i is the probability of a tuple x_i-y_i. Here for each pair x-y, there is only one z. This is known as *secondary quantization* – it corresponds to set-valued fuzzy techniques.

Physical analogues of type-2 fuzzy sets. Physicists like Wheeler have explored the natural possibility that the equations describing the fields are also varying, with some probability; see, e.g., [17]. Wheeler's logic may be unusual, but his logic is very straightforward.

According to Wheeler, one of the main ideas of quantum physics is that nothing is deterministic, everything is happening with some probability.

- In the original quantum theory, this idea is applied to quantities describing physical objects, such as coordinates, momentum, etc. In quantum physics we get slightly different results when we repeatedly measure the same quantity in several identical states. We cannot predict the measurement result, we can only predict the probability of different results.
- In the original quantum theory, while the quantities are probabilistic, the formulas describing the interaction between the objects – i.e., in physical terms, formulas describing the *fields* (like electromagnetic fields) – remain deterministic. The original theory described many physical phenomena, but in some cases, it turned out that the interaction is also only probabilistically determined. For example, with particles in the same locations, we also can have random deviations (*quantum fluctuations*) from the deterministic formulas. This is known as *secondary quantization*.
- In the secondary quantization, the probabilities of different field values are uniquely determined by current state of all the objects. Wheeler naturally conjectured that the formulas describing these probabilities can also change, and we can only know the probability of different formulas. This argument can be repeated again and again. As a result, what we can conclude that, in effect, any behavior is possible – just different behaviors occur with different probabilities.

So, we get quadruples x-y-z-t that correspond to type-2 fuzzy sets, etc.

Let us summarize these analogies in a single table.

uncertainty	physics
exact value	pre-quantum physics
set-valued uncertainty	free will
fuzzy uncertainty	quantum physics
Wheeler's ideas	type-2 fuzzy

4 How can this potentially speed up computations?

We need faster computers. While modern computers are very fast, there are many computational problems that need a drastic speed up. We need such a speed up to analyze global climate, to describe the effect of different medicines, etc. How can we speed up computations?

Speed of light – one of the main factors limiting computation speed. One of the main factors limiting computation speed is the speed of light 300 000 km/sec. Indeed, for the smallest laptop in which some parts are 30 cm away from each other, the fastest signal takes 30 cm / 300 000 km/sec = 1 nanosecond to go from one side of the computer to another. During this time, he usual 4 GHz processor performs 4 arithmetic operations.

Enter quantum effects. Due to the above simple calculation, to drastically speed up computations, we need to drastically decrease the size of the processor – i.e., drastically decrease the size of each computing cell, be it memory cell or processing cell. These cells are already very small. If we shrink them even more, we will reach sizes comparable to sizes of individual molecules – and in such sizes, it is necessary to take into account that such very small objects obey the laws of quantum physics. One of the ideas of quantum physics is that many quantities cannot take any possible real values, there is the smallest non-zero value – quantum. This is true for objects of all sizes, but:

- when the object is sufficient large – as in the our usual "macro"-world – the difference between this quantum and 0 can be safely ignored;
- however, when the object is small, of size comparable to the quantum size, the effects of quantum physics can no longer be ignored.

Quantum effects are already used to speed up computations. There are already algorithms that show that quantum effects can speed up computations; see, e.g., [20]:

- Grover's algorithms allows us to search for an element in an unsorted array of size n in time $\sqrt{n}$ – which is much faster than in non-quantum case, where in some cases, n steps are needed;
- Shor's algorithm can factorize large integers – and thus, decode most current encryptions – in feasible time, while the only known non-quantum algorithm, require astronomically un-realistic time for this.

Quantum effects can potentially speed up computations even further. As have mentioned, in quantum physics, everything becomes probabilistic, there are possible fluctuations of each field – and due to Heisenberg's uncertainty principle, the smaller the scale, the larger the fluctuations.

In particular, there are random fluctuations of the metric field – that described the local value of the speed of light. In particular, due to the Heisenberg's principle, the actual values of the speed of light vary – and the smaller the scale, the more they vary.

So, by following appropriate micro-paths where the speed of light is higher, we can send information faster – and the smaller spatial scale we use, the faster it can be.

Is there a physical evidence of this phenomenon? Is there evidence of this – or is it simply an untested theoretical conclusion? On the qualitative level, the evidence is all around us – it is the fact that:

- on the large scale, our Universe is largely unform, while
- due to the fast cosmological expansion, no speed-of-light mixing that could explain this uniformity is possible.

At presence, this uniformity is explained by so-called quantum inflation – which can be viewed as a particular example of the fact that in quantum physics, processes can spread faster than the non-quantum speed of light.

5 Remaining challenges

Challenges related to uncertainty. From the uncertainty viewpoint, it is desirable to analyze all the extended Z-number techniques that we outlined – so that these ideas can be applied in practical situations, where we hope that more accurately taking expert knowledge into account will lead to better results.

Challenges related to physics. From the viewpoint of physical reality, we face a more difficult challenge – how to incorporate freedom of will ideas into physical theories. This is challenging, but it will help us better understand our behavior and our decision-making.

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