B-matrices and their generalizations in the interval setting

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Abstract. This work focuses on generalizing a few easily recognizable subclasses of P-matrices into interval settings, including some results regarding these classes. Those classes are those of B-matrices, doubly B-matrices and B_π^R-matrices, for which we derive mainly means of recognition for interval variants, such as characterizations, necessary conditions and sufficient ones.

Keywords: B-matrix · Doubly B-matrix · B_π^R-matrix · Interval analysis · Interval matrix · P-matrix.

1 Introduction

P-matrices are defined as those matrices, whose all principal minors are positive, and they have close connection to so called linear complementarity problem, which is one of the reasons they are studied. A connection was even found between P-matrices and regularity of interval matrices, as shown in [2]. However, the task of verification whether a given matrix belongs to the class of P-matrices is co-NP-complete, as shown in [1]. This leads us to try to define some subclasses of P-matrices, that are easily recognizable. Such classes are e.g. B-matrices (introduced in [4]), doubly B-matrices (introduced in [5]) or B_π^R-matrices (introduced in [3]). What more, the B-matrices and doubly B-matrices found their use in localization of eigenvalues, as shown in [4] and [5].

In this work we will generalize our special subclasses of P-matrices into interval settings. We will lay grounds to recognizing the interval variants through characterization, be it through some property they posses or reduction, necessary conditions and sufficient ones.

First, let us note that by $\mathbb{IR}$ we denote the set of all real intervals and then let us take a look at what we mean by interval matrix.

Definition 1 (interval matrix). *An* interval matrix $\boldsymbol{A}$*, which we denote by* $\boldsymbol{A} \in \mathbb{IR}^{m\times n}$*, is defined as*

$$\boldsymbol{A} = \left[\underline{A}, \overline{A}\right] = \left\{ A \in \mathbb{R}^{m\times n} \middle| \underline{A} \le A \le \overline{A} \right\},$$

where $\underline{A}, \overline{A}$ *are called lower, respectively upper bound matrices of* $\boldsymbol{A}$*.*

We can as well look at $\boldsymbol{A}$ as matrix, which has entries from $\mathbb{IR}$, hence $\forall i \in [m], \forall j \in [n] : \boldsymbol{a}_{ij} = \left[\underline{a_{ij}}, \overline{a_{ij}}\right]$, where $[n] = \{1, 2, \ldots, n\}$ and analogously for $[m]$.

An interval matrix $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ is called an interval P-matrix if every $A \in \boldsymbol{A}$ is a P-matrix. Similarly we define interval B-matrices and interval doubly B-matrices. The only exception are B_π^R-matrices, where there is a certain ambiguity, leading to two natural extensions for interval matrices, which will be shown later.

2 B-matrices

Definition 2 (B-matrix). *Let $A \in \mathbb{R}^{n\times n}$. Then we say that A is a* B-matrix*, if $\forall i \in [n]$ the following holds:*

$$a) \quad \sum_{j=1}^{n} a_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \frac{1}{n}\sum_{j=1}^{n} a_{ij} > a_{ik}$$

Proposition 1. *B-matrices are P-matrices as well.*

The interval B-matrices are defined just as is mentioned above, but that definition gives us little to verify a matrix whether it is an interval B-matrix by, therefore we formulate the following characterization:

Theorem 1. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $\boldsymbol{A}$ is an interval B-matrix if and only if $\forall i \in [n]$ the following two properties hold:*

$$a) \quad \sum_{j=1}^{n} \underline{a}_{ij} > 0$$

$$b) \quad \forall k \in [n] \setminus \{i\} : \quad \sum_{j\neq k} \underline{a}_{ij} > (n-1)\overline{a}_{ik}$$

This characterization has time complexity $O(n^2)$, which is the same as the complexity of the characterization for real case from Definition 2.

3 Doubly B-matrices

Definition 3 (doubly B-matrix). *Let $A \in \mathbb{R}^{n\times n}$. Then we say that A is a* doubly B-matrix*, if $\forall i \in [n]$ the following holds:*

$$a) \quad a_{ii} > r_i^+$$

$$b) \quad \forall j \in [n] \setminus \{i\} : \left(a_{ii} - r_i^+\right)\left(a_{jj} - r_j^+\right) > \\ > \left(\sum_{k\neq i}\left(r_i^+ - a_{ik}\right)\right)\left(\sum_{k\neq j}\left(r_j^+ - a_{jk}\right)\right),$$

where $r_i^+ = \max\{0, a_{ij} | j \neq i\}$

Proposition 2. *Doubly B-matrices are P-matrices as well.*

Again the definition of interval doubly B-matrices gives us no tool to check if a given interval matrix belongs to the class of interval doubly B-matrices with, hence we introduce the following.

Theorem 2. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$. Then $\boldsymbol{A}$ is an interval doubly B-matrix if and only if the following two properties holds:*

a) $\forall i \in [n]: \quad \underline{a}_{ii} > \max\{0, \overline{a}_{ij} | j \neq i\}$ *and*
b) $\forall i, j \in [n], j \neq i, \forall k, l \in [n], k \neq i, l \neq j:$
(a) $\left(\underline{a}_{ii} - \overline{a}_{ik}\right) \cdot \left(\underline{a}_{jj} - \overline{a}_{jl}\right) >$
$$\left(\max\left\{0, \sum_{\substack{m\neq i\\ m\neq k}} \left(\overline{a}_{ik} - \underline{a}_{im}\right)\right\}\right) \cdot \left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)$$
(b) $$\underline{a}_{ii} \cdot \left(\underline{a}_{jj} - \overline{a}_{jl}\right) > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right) \cdot \left(\max\left\{0, \sum_{\substack{m\neq j\\ m\neq l}} \left(\overline{a}_{jl} - \underline{a}_{jm}\right)\right\}\right)$$
(c) $$\underline{a}_{ii} \cdot \underline{a}_{jj} > \left(\max\left\{0, -\sum_{m\neq i} \underline{a}_{im}\right\}\right) \cdot \left(\max\left\{0, -\sum_{m\neq j} \underline{a}_{jm}\right\}\right)$$

This characterization has time complexity $O(n^4)$, which is two orders of magnitude higher then for the real case, given the $O(n^2)$ complexity of the characterization from Definition 3.

4 $\mathbf{B}_{\boldsymbol{\pi}}^{\boldsymbol{R}}$-matrices

Definition 4 ($\mathbf{B}_{\pi}^{R}$-matrix). *Let $A \in \mathbb{R}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that*

$$0 < \sum_{j=1}^{n} \pi_j \leq 1$$

and let $R \in \mathbb{R}^n$ be a vector formed by the row sums of A. Then we say that A is a B_{π}^{R}-matrix, *if $\forall i \in [n]$:*

$$\begin{aligned} &a) \quad R_i > 0 \\ &b) \quad \forall k \in [n] \setminus \{i\}: \quad \pi_k \cdot R_i > a_{ik} \end{aligned}$$

Proposition 3. *B_{π}^{R}-matrices with $\pi \geq 0$ are P-matrices as well.*

As said earlier, there are two natural extensions for interval versions of B_{π}^{R}-matrices.

Definition 5 (homogeneous interval $\mathbf{B}_{\pi}^{\boldsymbol{R}}$-matrix). *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^n$. Then we say that $\boldsymbol{A}$ is a* homogeneous interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix, *if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}$ such that A is a (real) B_{π}^{R}-matrix.*

Definition 6 ((heterogeneous) interval $\mathrm{B}_{\Pi}^{\boldsymbol{R}}$-matrix). *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\boldsymbol{R} \in \mathbb{IR}^{n}$. Then we say that $\boldsymbol{A}$ is a* (heterogeneous) interval $\mathrm{B}_{\Pi}^{\boldsymbol{R}}$-matrix*, if $\forall A \in \boldsymbol{A}$: $\exists R \in \boldsymbol{R}, \exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$: A is a (real) B_{π}^{R}-matrix.*

However we found out that these two classes are, in a sense, the same.

Theorem 3. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval square matrix with positive row sums intervals. Then $\boldsymbol{A}$ is an interval $B_{\Pi}^{\boldsymbol{R}}$-matrix if and only if $\exists \pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and that $\boldsymbol{A}$ is a homogeneous interval $B_{\pi}^{\boldsymbol{R}}$-matrix.*

Hence we defined a class of interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrices.

Definition 7 (interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix). *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ and $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$. Then we say that $\boldsymbol{A}$ is an* interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix *if it is a homogeneous interval $B_{\pi}^{\boldsymbol{R}}$-matrix.*

And now once again we find ourselves in a position, where we lack any lead to how to test for the property of being an interval $\mathrm{B}_{\pi}^{\boldsymbol{R}}$-matrix in a given interval matrix. The following will help.

Theorem 4. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$, let $\pi \in \mathbb{R}^n$ such that $0 < \sum_{j=1}^{n} \pi_j \leq 1$ and $\boldsymbol{R} \in \mathbb{IR}^{n}$ be a vector of intervals of the individual row sums in matrix $\boldsymbol{A}$. Then $\boldsymbol{A}$ is an interval $B_{\pi}^{\boldsymbol{R}}$-matrix if and only if $\forall i \in [n]$ the following properties hold:*

a) $\underline{R}_i > 0$

b) $\forall k \in [n] \setminus \{i\}$:

$$\left(\pi_k > 1 \quad \Rightarrow \quad \sum_{j\neq k} \underline{a}_{ij} \ > \ \left(\frac{1}{\pi_k} - 1\right) \underline{a}_{ik}\right) \quad \wedge$$

$$\wedge \quad \left(0 < \pi_k \leq 1 \quad \Rightarrow \quad \sum_{j\neq k} \underline{a}_{ij} \ > \ \left(\frac{1}{\pi_k} - 1\right) \overline{a}_{ik}\right) \quad \wedge$$

$$\wedge \quad \left(\pi_k = 0 \quad \Rightarrow \quad 0 \ > \ \overline{a}_{ik}\right) \quad \wedge$$

$$\wedge \quad \left(\pi_k < 0 \quad \Rightarrow \quad \sum_{j\neq k} \overline{a}_{ij} \ < \ \left(\frac{1}{\pi_k} - 1\right) \overline{a}_{ik}\right)$$

This characterization has time complexity $O(n^2)$, which is, surprisingly, the same as a characterization from the definition of the real case, Definition 4 (although the interval case has undoubtedly higher implementational complexity).

Theorem 5. *Let $\boldsymbol{A} \in \mathbb{IR}^{n\times n}$ be an interval square matrix with positive row sums intervals (hence $\forall i \in [n]$: $\sum_{j=1}^{n} \underline{a}_{ij} > 0$). Then there exists a vector*

$\pi \in \mathbb{R}^n$ *satisfying* $0 < \sum_{j=1}^{n} \pi_j \leq 1$ *such that* $\boldsymbol{A}$ *is an interval* $B_\pi^{\boldsymbol{R}}$*-matrix if and only if*

$$\sum_{j=1}^{n} \max \left\{ \frac{\overline{a}_{ij}}{\overline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}}, \frac{\underline{a}_{ij}}{\underline{a}_{ij} + \sum\limits_{m \neq j} \underline{a}_{im}} \middle| \, i \neq j \right\} < 1.$$

Just as for the previous characterization, this one has time complexity $O(n^2)$ as well.

5 Concluding remarks

Here it is worth noting that both the interval doubly B-matrices and the interval $B_\pi^{\boldsymbol{R}}$-matrices are generalizations of interval B-matrices, just as the real doubly B-matrices and $B_\pi^{\boldsymbol{R}}$-matrices are generalizations of real B-matrices. Every interval (and analogically real) B-matrix is an interval (or real, respectively) doubly B-matrix and an interval (or real, respectively) $B_\pi^{\boldsymbol{R}}$-matrix for $\pi = \left(\frac{1}{n}, \ldots, \frac{1}{n}\right)$.

The last thing to mention is that, as can be seen, the characterization of interval doubly B-matrices has the time complexity two orders of magnitude higher than its real version in contrast with the B-matrices and B_π^R-matrices, thus for interval doubly B-matrices we focused more on finding necessary conditions and sufficient ones, whereas for interval B-matrices and B_π^R-matrices we delved more into their fundamental and closure properties.

References

1. Coxson, G.E.: The P-matrix problem is co-NP-complete. Math. Program. **64**(1), 173–178 (1994)
2. Hladík, M.: On relation between P-matrices and regularity of interval matrices. In: Bebiano, N. (ed.) Applied and Computational Matrix Analysis, Springer Proceedings in Mathematics & Statistics, vol. 192, pp. 27–35. Springer (2017)
3. Neumann, M., Peña, J., Pryporova, O.: Some classes of nonsingular matrices and applications. Linear Algebra Appl. **438**(4), 1936–1945 (2013)
4. Peña, J.M.: A class of P-matrices with applications to the localization of the eigenvalues of a real matrix. SIAM J. Matrix Anal. Appl. **22**(4), 1027–1037 (2001)
5. Peña, J.M.: On an alternative to Gerschgorin circles and ovals of Cassini. Numer. Math. **95**(2), 337–345 (2003)