Rubustness Properties of Absolute Value Linear Programming Problems and Relations to Interval Analysis

Milan Hladík[0000−0002−7340−8491]

Abstract The problem of solving absolute value systems and absolute value linear programming is a recent and developing area of research. It is formulated as an optimization problem in which the objective and constraint functions consist of linear terms and absolute values. Due to the presence of absolute values, the problem is NP-hard, even in very specific forms. Our aim is twofold. First, we review robust properties of the solution set, such as solvability and boundedness, where robustness refers to variations in the right-hand side vector. Second, we examine the relationship between this problem and interval linear programming. Indeed, it turns out that both problems share many similar aspects.

1 Introduction

Absolute value linear programming (AVLP). AVLP problems were introduced by Mangasarian [4] as optimization problems involving not only linear functions but also absolute values. Such problems as be expressed in the form

$$f = \max\ c^T x \ \text{ subject to }\ Ax - D|x| \leq b, \tag{1}$$

where $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and $A, D \in \mathbb{R}^{m\times n}$. Without loss of generality we can assume that D is nonnegative; the negative coefficients can be equivalently reformulated by linear constraints.

Absolute value problems have received considerable attention in recent years, particularly in connection with solving the absolute value equation $Ax + |x| = b$. In contrast, there are much fewer results concerning the optimization problem (1). Besides the pioneering work by Mangasarian [4], the problem was later studied by Yamanaka and Fukushima [10], Mangasarian [5], and Hladík and Hartman [2], among others. In partic-

Milan Hladík
Charles University, Faculty of Mathematics and Physics, Department of Applied Mathematics, Malostranské nám. 25, 11800, Prague, Czech Republic, e-mail: hladik@kam.mff.cuni.cz

ular, Yamanaka and Fukushima [10] proposed a branch-and-bound method for solving the problem, whereas Mangasarian [5] designed an unsupervised classification model based on absolute value linear programming. Hladík and Hartman [2] investigated fundamental properties of the solution set, including solvability, convexity, connectedness, boundedness, and the integrality of its vertices.

Complexity. From the computational point of view, AVLP problems are challenging. Many special versions of this model are NP-hard. For example, checking solvability of the system $Ax + |x| = b$ is NP-hard (see the survey paper [3]). As another example, integer linear programming is equivalent to AVLP by simple transformations.

The goal. Our goal is to address two related questions: first, the basic robust properties of the solution set; and second, the relationship to interval-valued linear programming problems [1].

2 Robust properties of the solution set

We present two robust properties of the solution set, for which we have a complete characterization; see Hladík and Hartman [2]. For the other properties, only partial results are known.

Solvability. It is known that a system of linear inequalities $Ax \le b$ is feasible for every b if and only if it is feasible for one, arbitrarily chosen, right-hand side vector b; for instance for $b := -e = -(1, \dots, 1)^T$.

An analogous result holds for the feasible set of AVLP, too. The feasible set of (1) is nonempty for every $b \in \mathbb{R}^m$ if and only if it is nonempty for $b := -e$ (or any other fixed negative vector). However, the analogy with linear systems is not valid with respect to the computational complexity: checking feasibility of (1) is NP-hard even on a class of problems with $b = -e$.

Boundedness. Another well-known problem in optimization is that a system of linear inequalities $Ax \le b$ is bounded for every b if and only if it is bounded for $b := 0$. Again, an analogous result holds for the feasible set of AVLP. The feasible set of (1) is bounded for each $b \in \mathbb{R}^n$ if and only if it is bounded for the particular choice of $b := 0$. However, this particular choice is still intractable – it is a co-NP-complete problem to check boundedness even when we a priori know that $b = 0$.

3 Relation to interval linear programming

The AVLP problems are closely related to linear programming (LP) problems with interval coefficients. We briefly mention several of these connections. For additional results, see Rohn [9], who lists forty different conditions for characterizing the regularity of an interval matrix. Many of these conditions rely on the solvability of systems of equations or inequalities involving absolute values.

Optimal value interpretation. Consider a class of LP problems

$$\max\ c^T x \ \text{ subject to }\ \tilde{A} \leq b, \tag{2}$$

where $\tilde{A} \in [A - D, A + D]$. Then f is equal to the best-case optimal value [1], that is,

$$f = \max_{\tilde{A}\in[A-D,A+D]} \max\ c^T x \ \text{ subject to }\ \tilde{A} \leq b.$$

Duality. Duality in linear programming is a very strong tool. The AVLP problems admit a sort of duality via the interval LP problems. Consider the class of LP problems

$$\min\ b^T y \ \text{ subject to }\ \tilde{A}^T y = c,\ y \geq 0, \tag{3}$$

where $\tilde{A} \in [A - D, A + D]$. Notice that (3) is the standard dual to (2). Now, under the assumption that the linear system

$$(A + D)u - (A - D)v \leq b,\ u, v \geq 0$$

is solvable, we get that f is its worst-case optimal value [1, 6, 8]. That is,

$$f = \max_{\tilde{A}\in[A-D,A+D]} \min\ b^T y \ \text{ subject to }\ \tilde{A}^T y = c,\ y \geq 0.$$

Theorem of the alternatives. Duality is also closely related to the theorem of the alternatives. We reformulate a result by Rohn [7] to emphasize the connection between interval analysis and absolute value systems. In the theorem below, the first condition involves strong solvability of an interval system, whereas the second condition involves an absolute value system.

Theorem 1 *Let $A, D \in \mathbb{R}^{n\times n}$, where $D \geq 0$. Then exactly one of the following alternatives holds:*

1. *the interval system*
$$Ax - [-D, D]|x| = b$$
is strongly uniquey solvable, that is, each realization has a unique solution;
2. *the inequality*
$$|Ax| \leq D|x|$$
has a nontrivial solution.

4 Conclusion

Some robust properties of the solution set of (1) can be derived as an analogy of the systems of linear inequalities. On the other hand, certain properties are not directly extendable. For example, connectedness or convexity are more difficult to characterize and no simple sufficient and necessary condition for them is know.

Also the connection of AVLP and the area of interval linear programming can be more explored. For interval LP problems, the theory and methods are quite developed [1], so it is quite promising to find more deeper relations between these areas.

Acknowledgment

The author was supported by the Czech Science Foundation Grant 25-15714S.

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