Solution to Homework Problem 22

Homework problem 22. Prove that the square root of 3 is not a rational number.

Solution. Let us prove it by contradiction. Let us assume that $\sqrt{3}$ is a rational number, i.e., $\sqrt{3} = a/b$ for some integers a and b.

If the numbers a and b have a common factor, then we can divide both a and b by this factor and get the same ratio. Thus, we can always find a and b that have no common factors.

Let us now get a contradiction.

- Multiplying both sides of the above equality by b, we get $\sqrt{3} \cdot b = a$.
- Squaring both sides, we get $3b^2 = a^2$.
- The left-hand side of this equality is divisible by 3, so the right-hand side $a^2 = a \cdot a$ must also be divisible by 3.
- Thus, a is divisible by 3, i.e., $a = 3p$ for some integer p.
- For $a = 3p$, we have $a^2 = (3p) \cdot (3p) = 9p^2$.
- Substituting $a^2 = 9p^2$ into the formula $3b^2 = a^2$, we get $3b^2 = 9p^2$.
- Dividing both sides by 3, we get $b^2 = 3p^2$.
- The right-hand side of this equality is divisible by 3, so the left-hand side $b^2 = b \cdot b$ must also be divisible by 3.
- Thus, b is divisible by 3.
- So, a and b have a common factor 3 – which contradicts to the fact that a and b have no common factors.

This contradiction shows that our original assumption – that $\sqrt{3}$ is a rational number – is wrong. The statement is proven.